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Topics in Ruin Theory: Optimal Reinsurance in a Context of Dependence, Analysis of the Gerber-Shiu Function with Reinsurance and Ruin Probabilities with Phase-type Distributions

Christophe Dutang

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Résumé

La théorie du risque est l'étude des problématiques (à court terme et long terme) d'un portefeuille d'assurance non-vie. Elle regroupe entre autres la théorie de la ruine et la réassurance. Cette dernière consiste à transférer tout ou partie d'un risque d'une assurance vers une autre. La théorie de la ruine, quant à elle, est l'analyse à long terme de la ruine d'une assurance (non-vie). L'étude des différentes mesures de ruine a été unifiée par la fonction Gerber-Shiu. Elle est définie comme la fonction actualisée de pénalité espérée et permet d'étudier les mesures de ruine, telle que la probabilité de ruine. Trois sujets sont abordés dans ce mémoire, avec comme points communs la réassurance et la théorie de la ruine. Le premier chapitre se concentre sur la réassurance optimale, lorsqu'on utilise le coefficient d'ajustement comme mesure de ruine. On montre que le coefficient d'ajustement est une fonction unimodale du paramètre de rétention, le tout dans un modèle de dépendance entre le coût et l'arrivée des sinistres. Le deuxième chapitre utilise la fonction de Gerber-Shiu dans le modèle de Cramér-Lundberg lorsqu'on inclut de la réassurance proportionnelle. Enfin, le dernier chapitre traite du calcul de la probabilité de ruine à l'aide des lois phase-type dans le modèle de Sparre Andersen. En supposant des temps d'attente et des montants de sinistres de loi phase-type, on obtient des expressions explicites de la probabilité de ruine avec une réassurance proportionnelle. L'implémentation des calculs a été intégrée au package R **actuar**.

Mots-clés : **Coefficient d'ajustement** ; **Coefficient de Lundberg** ; **Copules** ; **Théorie du risque** ; **Modèles avec dépendencs** ; **Réassurance proportionnelle** ; **Réassurance excess of loss** ; **Réassurance optimale** ; **Loi phase-type** ; **Fonction Gerber-Shiu**

Abstract

Risk theory can be defined as the non-life insurance mathematics. Ruin theory and reinsurance are parts of risk theory, which study respectively the long-term ruin of an insurance company and the risk transfer from one insurance company to another. The analysis of ruin measures had been unified by the Gerber-Shiu function, which allows us to study ruin measures such ruin probability. We study three different topics, whose overall subjects are reinsurance and ruin theory. The first chapter focuses on optimal reinsurance, when we use the adjustment coefficient as a ruin measure. In a context of dependence between claim severity and claim frequency, we show the adjustment coefficient is a unimodal function of the retention parameter, either for proportional or excess of loss reinsurance. Chapter 2 deals with the Gerber-Shiu function with proportional reinsurance in the well-known Cramér-Lundberg model. Finally, we give our attention on the computation of the ruin probability thanks to phase-type distributions in the Sparre Andersen model. We derive explicit ruin probabilities, when assuming both claim sizes and inter-occurence times are phase-type distributed. These computation has been inserted into the R package **actuar**.

Keywords : **Adjustment coefficient**; **Lundberg coefficient**; **Copula**; **Ruin theory**; **Dependence models**; **Proportional reinsurance**; **Excess of loss reinsurance**; **Optimal reinsurance**; **Phase-type distributions**; **Gerber-Shiu function**

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∗. the actuarial lab of Université Laval

Contents

Introduction

Risk theory studies all the aspects of a non-life insurance portfolio. In this wide area, ruin theory focuses on the long term ruin of an insurance company with a such portfolio. Another part of risk theory deals with the process of reinsurance, in which an insurance company transfers part or all its risk to another insurance company (called the reinsurer). Reinsurance and ruin theory are parts of risk theory which are closely related. Ruin theory is studied since more than a century. At the beginning of the XX$^{\text{th}}$ century, Swedish actuaries Lundberg and Cramér created the fundamentals of the classical continuous-time risk model (where claim arrival process is assumed to be a Poisson process). This model had been widely extended during the last century. Andersen deeply improved the Cramér-Lundberg model in 1957 when he considered the claim arrival process to be a renewal process. Recently, Gerber & Shiu (1998) revisited the ruin theory with their expected discounted penalty function. The so-called Gerber-Shiu function allows us to analyse ruin measures such as the ruin probability, the behavior of the surplus at ruin, etc... Their works gived new insights into ruin theory.

Since the mid nineties, models with dependence have been the interest of many researchers. For instance, the work of Albrecher & Teugels (2006) deals with ruin probability when claim severity and claim frequency are dependent. Other kinds of dependence have been studied such as two dependent lines of business in a portfolio and claim severity and claim frequency dependent on a common intensity variable.

Recent studies also concentrated on optimal reinsurance, whose aim is to choose the best reinsurance according to a certain criterion. Waters (1983) and Centeno (2002*b*) use the adjustment coefficient as a risk measure to choose optimal reinsurance, either with proportional reinsurance or excess of loss reinsurance. They work in the Sparre Andersen model, where independence is assumed between claim sizes and inter-occurence times.

Though phase-type distributions are known since nearly a century, its application in ruin theory dates from the nineties. Phase-type distributions are a wide class of positive random variable distributions, in which there are among others the exponential distribution, the Erlang distribution and the hyper-exponential distribution. Asmussen (1992) presents the advantages to use phase-type distributions to compute ruin probabilities in the Sparre Andersen model. He showed the ruin probabilities have very easy (explicit) expressions when claim sizes are phase-type distributed.

This research memoir is divided into three independent chapters, but with common topics: reinsurance and ruin theory. The first chapter extends the work of Centeno (2002*b*) by assuming claim sizes and inter-occurence times are no longer independent. We use the adjustment coefficient to find optimal reinsurance in a context of dependence between claim severity and claim frequency.

In the second chapter, we introduce reinsurance directly into the surplus process. We study in details the Gerber-Shiu function in the proportional reinsurance case. In the model of Cramér-Lundberg with reinsurance, we derive many ruin-related quantities.

The third chapter concentrates on usage of phase-type distributions to obtain explicit ruin probabilities. First, we present results on the effect of proportional reinsurance in the Sparre Andersen model. Second, we implement ruin probability computations in the R package **actuar**. The memoir then concludes with a discussion of possible further research.

Chapter 1

Optimal Reinsurance in a Context of Dependence

In several studies on optimal reinsurance, the assumption of independence between claim sizes and inter-occurence times facilitates the results deducted from the models. Many works has assumed the case of independence on maximising the adjustment coefficient such as Waters (1983), Centeno (2002*a*), Centeno (2002*b*) and Hald & Schmidli (2004). Centeno (1995) also deals with optimal reinsurance (again in the case of independence) of the finite time ruin probability, when mitigating a more sophisticated bound * of this ruin probability.

When dependence between claim sizes and inter-occurence times is made, the studied ruin models seldom focus on reinsurance (e.g. Albrecher & Teugels (2006) , Boudreault et al. (2006) and Marceau (2007)). Only the work of Centeno (2005) deals with dependence in a context of optimal reinsurance (excess of loss precisely), where the dependence is characterized through the claim frequency.

So the study of optimal reinsurance in a context of dependence comes naturally. First, we give our attention on optimal reinsurance retention level in a context of dependence, when the premium is calculated according to the expected value principle at first, and then with other premium calculation principles.

In this chapter, we consider a general risk model with $(N_t)_{t\in\mathbb{R}^+}$, the renewal process of number of claims (i.e. N_t can be written as $\sup(n \in \mathbb{N}, T_n \leq t)$ with $T_0 = 0,\ T_n = \sum_{i=1}^n W_i$) and $(X_i)_{i\in\mathbb{N}^\star}$, the sequence of claim sizes. We assume that the couple of inter-occurence times and claim sizes, $(W_i, X_i)_{i\in\mathbb{N}^\star}$, forms a sequence of independent and identically distributed (strictly) positive random variables. If claim sizes X_i and waiting time W_i were assumed independent, this would be the Sparre Andersen model. Then we define the ruin time of the insurance company as the first time where the insurance surplus is negative

$$\tau_u = \inf(t > 0, u + Ct - S_t < 0),$$

where u denotes the initial surplus, C the premium rate and S_t the total claim amount at time t (i.e. $S_t = \sum\limits_{i=1}^{N_t} X_i$).

*. sometimes called the Gerber's bound, cf. pp 139 of Gerber (1979)

If ruin does not occur, $\tau_u = +\infty$. The premium rate C must satisfy the following condition, so as to avoid the ruin almost surely: $E[X - CW] < 0$, which is equivalent to

$$C = (1+\eta)\frac{E[X]}{E[W]},$$

where $\eta > 0$ is the safety loading. It is well known that the adjustment coefficient R, which verifies the equation $E\left[e^{r(X-CW)}\right] = 1$, provides an exponential bound to the infinite time ruin probability $\psi(u)$:

$$\psi(u) \stackrel{\triangle}{=} P(\tau_u < +\infty) \leq e^{-Ru}.$$

Thus, the ruin probability is controlled by the adjustment coefficient R (i.e. the adjustment coefficient is a measure for the risk).

The main objective of this part is to present optimal reinsurance, which consists in maximising the adjustment coefficient, with two kinds of reinsurance: proportional and excess of loss reinsurance. Unlike previous works in this area, we work in a context of dependence between X and W (resp. claim sizes and inter-occurence times), where the expected value premium calculation principle is applied *. Therein, we prove that the adjustment coefficient R is a unimodal of function of the retention levels, in general for proportional reinsurance and under a specific assumption for excess of loss reinsurance.

There are various ways to integrate dependence. Firstly, we use copulas to structure the dependence between claim size and claim frequency. From this approach, the issue of unimodality will be studied in some "extreme" cases of dependence. Secondly, we will focus on two particular cases of dependence: one, where the dependence is made on the conditional distribution of claim sizes; and the other, where we use a common frailty approach on claim size and frequency distribution.

This chapter is divided into seven sections. In section 1.1, we will study the proportional reinsurance case, whereas the section 1.2 focuses on excess of loss reinsurance. As the first two parts give only theoretical results, numerical applications are carried out in section 1.3, when the dependence is modelled through copulas. Then, in section 1.4, we will analyze three special cases of dependence between X and W : comonotonic, independent and countermonotonic. Finally, section 1.5 and 1.6 present a conditional and a common frailty structure of dependence. The last section concludes.

1.1 Proportional reinsurance

In this section, we focus on proportional reinsurance. The net (of reinsurance) annual claims $X(a)$ is defined as aX (i.e. $a \in]0,1]$ is the retention rate). Given a retention rate, the net premium per unit of time is expressed as follows

$$C(a) = \underbrace{(1+\eta)\frac{E[X]}{E[W]}}_{\text{insured risk}} - \underbrace{(1+\eta_R)\frac{E[(1-a)X]}{E[W]}}_{\text{reinsured risk}}, \tag{1.1}$$

$*$. at first, then other premium principles will be considered.

where η and η_R denote the risk margin (supposed known and constant) respectively for the insurer and the reinsurer. The risk margins satisfy the condition $\eta < \eta_R$, otherwise the insurer could get rid of all his risk by insuring his whole portofolio. The premium rate defined in equation (1.1) can be expressed in a simplified form :

$$C(a) = \frac{E[X]}{E[W]}(\eta - \eta_R + a(1 + \eta_R)).$$

Let us notice this premium is a linear function of the retention rate a. So the derivative of the premium, $C'(a)$, is constant : $\frac{E[X]}{E[W]}(1+\eta_R)$. Note that the premium rate $C(a)$ is not always positive, this will be discussed in the following sub-section.

We are concerned with optimal reinsurance in context of dependence between claim sizes and claim inter-occurence times. So we look for the optimal retention rate $a^\star$ which maximizes the adjusment coefficient R. The adjustment coefficient R is the unique positive root of the following equation

$$E\left[e^{r(X(a)-C(a)W)}\right] = 1, \tag{1.2}$$

which is equivalent to

$$h(r, a) = \ln\left(E\left[e^{r(X(a)-C(a)W)}\right]\right) = 0. \tag{1.3}$$

We use the equation (1.3) rather than (1.2) because it eases the analysis of the adjustment coefficient.

1.1.1 Admissibility condition on 'a'

First, let us consider the condition on the retention rate a so that the equation (1.3) has a strictly positive root, the adjustment coefficient. The partial derivative of h with respect to r is given by

$$\frac{\partial h}{\partial r}(r, a) = \frac{E\left[(aX - C(a)W)e^{r(X(a)-C(a)W)}\right]}{E\left[e^{r(X(a)-C(a)W)}\right]}.$$

Since the function $r \mapsto h(r, a)$ is convex (cf. appendix A.1) and $h(0, a) = 0$, the root of equation (1.3) exists if and only if $\frac{\partial h}{\partial r}(0, a) < 0$ *. Let g be $a \mapsto \frac{\partial h}{\partial r}(0, a)$, the first derivative of h with respect to r as a function of a

$$g(a) = E\left[aX - C(a)W\right].$$

We must find the values of a where $g(a)$ is strictly negative. As the function g is a (strictly) decreasing function ($g'(a) = -\eta_R E[X] < 0$), g has at most one root. The equation $g(a) = 0$ is equivalent to

$$aE[X] - C(a)E[W] = 0,$$

which yields to †

$$a = \frac{\eta_R - \eta}{\eta_R}.$$

*. otherwise the function $r \mapsto h(r, a)$ is a strictly increasing convex function. And the only root of (1.3) is 0.
†. cf. appendix A.2

Let a_0 be $\frac{\eta_R-\eta}{\eta_R}$, which is positive since $\eta < \eta_R$. Therefore $\forall a \in]a_0, 1]$, $g(a) < 0$ that is to say that it exists $R > 0$ such that , $h(R, a) = 0$. Otherwise, the root of (1.3) is null. Furthermore, the premium rate $C(a)$ is strictly positive on $]a_0, 1]$, since the condition $g(a) < 0$ is exactly the net profit constraint, assumed to avoid the certain ruin.

1.1.2 Unimodality of R(a)

Let us study the optimal adjustment coefficient. From the previous subsection, we already know that the adjustment coefficient R exists if and only if $a \in]a_0, 1]$. In the rest of this section, we suppose that $a \in]a_0, 1]$.

Unimodal functions

We recall the definition of a unimodal function ϕ on I.

Definition. *$\phi : t \mapsto \phi(t)$ is a unimodal function on I if ϕ has a unique maximum reached for $t = t^\star$ on I and ϕ is a strictly increasing function on $I \cap] - \infty, t^\star]$ and a strictly decreasing function on $I \cap]t^\star, +\infty]$.*

The function ϕ can also be called unimodal if it is first strictly decreasing and then strictly increasing (i.e. ϕ has a unique minimum on I), but this is not the case we study here. Furthermore, we have the following sufficient condition * of unimodality,

Proposition. *If ϕ is a C^2 function, ϕ is a unimodal function on I if the equation $\phi'(t) = 0$ has a unique root $t^\star$, such as $\phi''(t^\star) < 0$.*

To prove that the retention function $R(a)$ is unimodal, we show that this function verifies the previous sufficient condition. Firstly, we prove that the equation $\frac{\partial R}{\partial a}(a) = 0$ has a unique root $a^\star$. Then, we show that $\frac{\partial^2 R}{\partial a^2}(a^\star) < 0$.

Part 1

Using the implicit function theorem †, we get

$$\frac{\partial R}{\partial a}(a) = - \left. \frac{\frac{\partial h}{\partial a}(r, a)}{\frac{\partial h}{\partial r}(r, a)} \right|_{r=R} . \tag{1.4}$$

This theorem requires the denominator to be non null. Indeed, we already know that $r \mapsto h(r, a)$ is a convex function, since $\frac{\partial^2 h}{\partial r^2}(r, a) < 0$ ‡. So, the latter function has a unique minimum on $\tilde{r}$, such that $h(\tilde{r}, a) < 0$ since $h(0, a) = 0$ and $\frac{\partial h}{\partial r}(0, a) = E[X(a) - C(a)W] < 0$. Therefore, the adjustment

*. cf. proof in appendix A.3
†. recalled in appendix A.4
‡. cf. appendix A.1

coefficient R verifies $R > \tilde{r}$. Thus, we can conclude $\forall a > 0$, $\frac{\partial h}{\partial r}(R, a) > 0$ since $r \mapsto h(r, a)$ is an increasing function on $[\tilde{r}, +\infty[$.

In consequence, the equation $\frac{\partial R}{\partial a}(a) = 0$ is equivalent to

$$\left.\frac{\partial h}{\partial a}(r, a)\right|_{r=R} = 0. \tag{1.5}$$

Let us verify that the equation (1.5) has a unique root $a^\star$. The equation (1.5) is equivalent to

$$\frac{E\left[R(X - C'(a)W)e^{R(X(a)-C(a)W)}\right]}{E\left[e^{R(X(a)-C(a)W)}\right]} = 0,$$

which yields to

$$E\left[(X - C'(a)W)e^{R(X(a)-C(a)W)}\right] = 0,$$

since $R > 0$ and $E\left[e^{R(X(a)-C(a)W)}\right] > 0$. Let f be the function $a \mapsto E\left[(X - C'(a)W)e^{R(X(a)-C(a)W)}\right]$, defined as the left-hand side of the previous equation. As shown in appendix A.5, f has a unique root. Note that, we have

$$f(a_0) = -\eta_R E[X] < 0 \quad \text{and} \quad f(1) > 0\,^*,$$

Hence, f cancels exactly once on $]a_0, 1]$, i.e. the equation (1.5) has a unique root $a^\star$.

Part 2

Now let us find the sign of the second derivative $\frac{\partial^2 R}{\partial a^2}$ at the optimal retention rate $a^\star$. From 1.4, the second derivative of R can be easy calculated when the first derivative is null. We get

$$\frac{\partial^2 R}{\partial a^2}(a^\star) = -\left.\frac{\frac{\partial^2 h}{\partial a^2}(r, a)}{\frac{\partial h}{\partial r}(r, a)}\right|_{r=R, a=a^\star}.$$

The numerator is given by

$$\frac{\partial^2 h}{\partial a^2}(R, a^\star) = \frac{E\left[R(X - C'(a^\star)W)^2 e^{R(X(a^\star)-C(a^\star)W)}\right]}{E\left[e^{R(X(a^\star)-C(a^\star)W)}\right]} - \left(\frac{E\left[R(X - C'(a^\star)W)e^{R(X(a^\star)-C(a^\star)W)}\right]}{E\left[e^{R(X(a^\star)-C(a^\star)W)}\right]}\right)^2.$$

Since $a^\star$ cancels the first derifative of R (hence the second member of the right-hand side), this yields to

$$\frac{\partial^2 h}{\partial a^2}(R, a^\star) = \frac{E\left[R^2(X - C'(a^\star)W)^2 e^{R(X(a^\star)-C(a^\star)W)}\right]}{E\left[e^{R(X(a^\star)-C(a^\star)W)}\right]}. \tag{1.6}$$

Hence, we have $\frac{\partial^2 h}{\partial a^2}(R, a^\star) > 0$. As a consequence, we have that the second derivative $\frac{\partial^2 R}{\partial a^2}(a^\star)$ has opposite sign as $\frac{\partial h}{\partial r}(R, a^\star)$, which is positive as we have already seen. Thus, $\frac{\partial^2 R}{\partial a^2}(a^\star) < 0$, that is to say the function $a \mapsto R(a)$ is unimodal on $]a_0, 1]$, as the function $a \mapsto \frac{\partial R}{\partial a}(a)$ cancels exactly once.

∗. cf. appendix A.5

Conclusion on unimodality

To conclude on this optimatility issue of $a \mapsto R(a)$, we have that the adjustment coefficient R in the case of proportional reinsurance is unimodal function of a on $]a_0, 1]$. Note that unimodality (sufficient condition for maximization) ensures that numerical maximizations of R will converge, which is particularly useful in practice.

1.1.3 Using other premium calculation principles

Until now, we have studied the adjustment coefficient R, when the premium are calculated according to the expected value principle. Let us study the following premium calculation principles:
- variance premium principle : $C = E[X] + \eta Var[X]$;
- standard deviation premium principle : $C = E[X] + \eta\sqrt{Var[X]}$;
- exponential premium principle : $C = \frac{\ln\left(E[e^{\eta X}]\right)}{\eta}$.

These premium principles are defined without reinsurance for an annual risk X. More details on their properties can be found in the Encyclopedia of Actuarial Science of Teugels & Sundt (2006). Let us study those premiums with proportional reinsurance with a retention rate a (as usual with $\eta < \eta_R$ the loading coefficients).

The differences in the demonstration of unimodality of $R(a)$ between the expected value premium principle and other premium principles appear (1) in the function $g(a)$, (whose sign makes R exist or not); (2) the function $f(a)$ (whose number of roots is the number of (local) maxima) and (3) the second derivative $\frac{\partial^2 h}{\partial a^2}(R, a^\star)$ of $R(a)$ (whose sign ensures the optima to be maxima or minima). For all premium principles, we have to study these three points.

Variance premium principle

The variance premium principle with proportional reinsurance is defined as follows

$$C(a) = \frac{E[X] + \eta Var[X]}{E[W]} - \frac{E[X(1-a)] + \eta_R Var[X(1-a)]}{E[W]} = \frac{aE[X] + Var[X](\eta - (1-a)^2\eta_R)}{E[W]}.$$

The derivatives of $C(a)$ are

$$C'(a) = \frac{E[X] + 2(1-a)\eta_R Var[X]}{E[W]} \text{ and } C''(a) = \frac{-2\eta_R Var[X]}{E[W]} < 0.$$

First, we need to study the admissibility condition on the retention rate, so that the adjusmten coefficient $R(a)$ exists. As in the previous sub-section, we defined the function $g(a) = E\left[aX - C(a)W\right]$. It can be expressed as

$$g(a) = -Var[X](\eta - (1-a)^2\eta_R).$$

Since $g(0) = -(\eta - \eta_R)Var[X] > 0$, g is strictly decreasing* convex function, g has a unique positive root a_0 on $[0, 1]$, such as $\forall a > a_0, g(a) < 0$. In this case, we have an explicit expression of $a_0 = 1 - \sqrt{\frac{\eta}{\eta_R}} > 0$. Thus, the adjustment coefficent $R(a)$ exists on $]a_0, 1]$.

*. $g'(a) = -2(1-a)\eta_R Var[X] \leq 0$ and $g''(a) = 2\eta_R Var[X]$.

Secondly, we have the following expression for 'f'

$$f(a) = E\left[(X - C'(a)W)e^{R(X(a)-C(a)W)}\right],$$

with $C'(a) = \frac{E[X]+2(1-a)\eta_R Var[X]}{E[W]}$. Differentiating f, we have

$$f'(a) = RE\left[(X - C'(a)W)^2 e^{R(X(a)-C(a)W)}\right] - C''(a)E\left[e^{R(X(a)-C(a)W)}\right] + R'(a)E\left[(X - C'(a)W)(X(a) - C(a)W)e^{R(X(a)-C(a)W)}\right].$$

Since $C''(a) < 0$, $f'(a)$ is strictly positive when f nullifies ($f(a) = 0 \Leftrightarrow R'(a) = 0$). Furthermore, we have

$$f(a_0) \stackrel{\triangle}{=} E\left[(X - C'(a_0)W)e^0\right] = -2(1-a_0)Var[X] < 0,$$

and

$$f(1) = E\left[(X - C'(1)W)e^{R(X-C(1)W)}\right] > \frac{E\left[(X - C(1)W)e^{R(X-C(1)W)}\right]}{E\left[e^{R(X-C(1)W)}\right]} > 0.$$

Indeed, we have when $a = 1$

$$C'(1) = \frac{E[X]}{E[W]} < \frac{E[X] + \eta Var[X]}{E[W]} = C(1).$$

So $f(1)$ is minorated by $\frac{\partial h}{\partial r}(R,a)\big|_{a=1}$, which is postive as we have already seen in the previous sub-section. Therefore, f is a continuous function, which starts from $f(a_0) < 0$ to $f(1) > 0$ and is a strictly increasing function, when f nullifies. Hence, f cancels once, say $a^\star$.

Finally, the second derivative of R (when the first one cancels) has the opposite sign of

$$\frac{\partial^2 h}{\partial a^2}(R, a^\star) = \frac{E\left[R(R(X - C'(a^\star)W)^2 - C''(a)W)e^{R(X(a^\star)-C(a^\star)W)}\right]}{E\left[e^{R(X(a^\star)-C(a^\star)W)}\right]}.$$

Note that the equation (1.6) is no longer verified since $C''(a) \neq 0$. But as $C''(a) < 0$, we have that $\frac{\partial^2 h}{\partial a^2}(R, a^\star) > 0$, hence $R''(a^\star) < 0$. So we can conclude the adjustment coefficient $R(a)$ is still unimodal on $]a_0, 1]$ with the variance premium calculation principle.

Standard deviation premium principle

The standard deviation premium principle with proportional reinsurance is given by

$$C(a) = \frac{E[X] + \eta\sqrt{Var[X]}}{E[W]} - \frac{E[X(1-a)] + \eta_R\sqrt{Var(X(1-a))}}{E[W]} = \frac{aE[X] + \sqrt{Var[X]}(\eta - (1-a)\eta_R)}{E[W]}.$$

The derivatives of $C(a)$ are

$$C'(a) = \frac{E[X] + \eta_R\sqrt{Var[X]}}{E[W]} \quad \text{and} \quad C''(a) = 0.$$

Let us study the 'g' function with this premium principle. g is given by

$$g(a) = -\sqrt{Var[X]}(\eta - (1-a)\eta_R),$$

which is a strictly decreasing* function on $[0,1]$. Thus there is a unique $a_0 \in]0,1[$, such that $\forall a > a_0, g(a) < 0$. Here, an explicit expression of the root can be found: $a_0 = \frac{\eta_R - \eta}{\eta_R} > 0$. Hence, the adjustment coefficient $R(a)$ exists on $]a_0, 1]$.

The study of the 'f' function is very similar as for the expected value principle, since $C''(a) = 0$. Indeed, we have $f(a_0) = -\eta_R\sqrt{Var[X]} < 0$. $f(1)$ is given by

$$f(1) = E\left[(X - C'(1)W)e^{R(X-C(1)W)}\right] > \frac{E\left[(X - C'(1)W)e^{R(X-C'(1)W)}\right]}{E\left[e^{R(X-C'(1)W)}\right]},$$

using

$$C'(1) = \frac{E[X] + \eta_R\sqrt{Var[X]}}{E[W]} > \frac{E[X] + \eta\sqrt{Var[X]}}{E[W]} = C(1).$$

Thus, $f(1)$ is minorated by $\frac{\partial h}{\partial r}(R,a)\Big|_{a=1} > 0$ †, if we consider that $R(1)$ (no reinsurance) is calculated with a loading coefficient η_R ‡. Therefore, f cancels once on $]a_0, 1]$,

And finally, the second derivative of R is negative, since the equation (1.6) is still verified ($C''(a) = 0$). Thus, we conclude that the adjustment coefficient $R(a)$ is still unimodal on $]a_0, 1]$ with the standard deviation premium calculation principle.

Exponential premium principle

The exponential premium principle with proportional reinsurance is given by

$$C(a) = \frac{\ln\left(E\left[e^{\eta X}\right]\right)}{\eta E[W]} - \frac{\ln\left(E\left[e^{\eta_R(1-a)X}\right]\right)}{\eta_R E[W]}.$$

The derivatives of $C(a)$ are

$$C'(a) = \frac{E\left[Xe^{\eta_R(1-a)X}\right]}{E[W]E\left[e^{\eta_R(1-a)X}\right]},$$

and

$$C''(a) = \frac{-\eta_R}{E[W]}\left[\frac{E\left[X^2 e^{\eta_R(1-a)X}\right]}{E\left[e^{\eta_R(1-a)X}\right]} - \left(\frac{E\left[Xe^{\eta_R(1-a)X}\right]}{E\left[e^{\eta_R(1-a)X}\right]}\right)^2\right].$$

We have $C''(a) < 0$ since the term between brackets is strictly positive because it is a variance of an Esscher transform.

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*. $g'(a) = -\sqrt{Var[X]}\eta_R < 0$.

†. cf. previous subsection

‡. $C'(1)$ is equivalent to the premium $\tilde{C}(1)$ with a loading coefficient η_R. Hence $\frac{E\left[(X-C'(1)W)e^{R(X-C'(1)W)}\right]}{E\left[e^{R(X-C'(1)W)}\right]} = \frac{E\left[(X-\tilde{C}(1)W)e^{R(X-\tilde{C}(1)W)}\right]}{E\left[e^{R(X-\tilde{C}(1)W)}\right]} = \frac{\partial \tilde{h}}{\partial r}(R,a)\Big|_{a=1}$

From the previous equations, we get

$$g(a) = E[X] - \frac{1}{\eta}\ln\left(E\left[e^{\eta X}\right]\right) + \frac{1}{\eta_R}\ln\left(E\left[e^{\eta_R(1-a)X}\right]\right).$$

Thus

$$g'(a) = E[X] - \frac{E\left[Xe^{\eta_R(1-a)X}\right]}{E\left[e^{\eta_R(1-a)X}\right]} = -\frac{Cov(X, e^{\eta_R(1-a)X})}{E\left[e^{\eta_R(1-a)X}\right]},$$

which is negative *. Furthermore, we have

$$g(0) = \frac{1}{\eta_R}\ln\left(E\left[e^{\eta_R X}\right]\right) - \frac{1}{\eta}\ln\left(E\left[e^{\eta X}\right]\right) > 0 \text{ and } g(1) = E[X] - \frac{1}{\eta}\ln\left(E\left[e^{\eta X}\right]\right) < 0,$$

since the exponential premium principle is an increasing function of the loading coefficient η, and it verifies the positive risk loading constraint. Therefore, g is a decreasing function, which nullifies once on $[0,1]$, say a_0. So, the adjustment coefficient $R(a)$ exists on $]a_0, 1]$ with $a_0 > 0$.

Let us study the 'f' function. We recall that f is defined as $E\left[(X - C'(a)W)e^{R(aX - C(a)W)}\right]$. We have

$$f(a_0) = E\left[(X - C'(a_0)W)e^0\right] = g'(a_0) < 0,$$

and

$$C'(1) = \frac{E[X]}{E[W]} < \frac{1}{\eta}\ln\left(E\left[e^{\eta X}\right]\right) = C(1),$$

by using the Jensen inequality with $\varphi(x) = e^{\eta x}$. Hence, we also have $f(1) > 0$. Using the same argument as the one used for the variance premium principle (where $C''(a) < 0$), f has a unique root $a^\star$, and so the first derivative $R'(a)$. Again, we used what was done for the variance premium principle, i.e.

$$\frac{\partial^2 h}{\partial a^2}(R, a^\star) = \frac{E\left[R(R(X - C'(a^\star)W)^2 - C''(a)W)e^{R(X(a^\star) - C(a^\star)W)}\right]}{E\left[e^{R(X(a^\star) - C(a^\star)W)}\right]},$$

which is positive because of $C''(a) < 0$. Hence, the second derivative of $R(a)$ is negative. And so, the adjustment coefficient is a unimodal function on $]a_0, 1]$ with the exponential premium principle.

1.2 Excess of loss reinsurance

This section is the analog of the previous section, when the insurer takes excess of loss reinsurance. Let $L \in \mathbb{R}^+$ be the retention limit of the insurer. Once reinsured, the insures keeps the risk $X(L) = X \wedge L = \min(X, L)$. As in the previous section, the risk margins η and η_R are known and constant. The net premium per unit of time $C(L)$ is expressed as follows:

$$C(L) = \underbrace{(1+\eta)\frac{E[X]}{E[W]}}_{\text{insured risk}} - \underbrace{(1+\eta_R)\frac{E\left[(X-L)_+\right]}{E[W]}}_{\text{reinsured risk}}, \tag{1.7}$$

*. cf. appendix A.6

where we again assume that $\eta < \eta_R$. The derivatives of C are given by

$$C'(L) = (1+\eta_R)\frac{\overline{F}_X(L)}{E[W]}{}^{*} \quad \text{and} \quad C''(L) = -(1+\eta_R)\frac{f_X(L)}{E[W]},$$

when the density f_X exists and $\overline{F}_X$ stands for the survival function of random variable X.
Our main focus is to maximize the adjustement coefficient R, which is the root of the well known equation

$$h(r,L) = \ln\left(E[e^{r(X(L)-C(L)W)}]\right) = 0, \tag{1.8}$$

which we call the adjustment coefficient equation (even if in the literature, the adjustment coefficient equation refers to $E[e^{r(X(L)-C(L)W)}] = 1$).

1.2.1 Admissibility condition on 'L'

Consider the function g defined as

$$L \mapsto \frac{\partial h}{\partial r}(0,L) = E[X \wedge L - C(L)W].$$

The adjustment coefficient equation (1.8) has a positive root if and only if $g(L) < 0$ (i.e. $\frac{\partial h}{\partial r}(0,L) < 0$), because of the same reason as in the case of proportional reinsurance (i.e. convexity of $r \mapsto h(r,L)$ in appendix A.1). The function g can be expressed in the following form

$$g(L) = (\eta_R - \eta)E[X] - \eta_R E[X \wedge L],$$

where the limited expected value $E[X \wedge L]$ is equal to $\int_0^L \overline{F}_X(x)dx$. g is a strictly decreasing function since $g'(L) = -\eta_R\overline{F}_X(L) < 0$ †. As

$$g(0) = (\eta_R - \eta)E[X] > 0 \text{ and } g(L) \underset{L\to+\infty}{\longrightarrow} -\eta E[X] < 0,$$

it exists $L_0 > 0$ which nullifies the function g. That is to say, we are ensured that there is $L_0 > 0$ such that $\forall L > L_0, g(L) < 0$. This finishes the proof, that the equation (1.8) has a positive root when $L \in \,]L_0, +\infty[$. Numerically, we found that L_0 is equal to 0.4054 and 0.3544 respectively when $X \sim \mathcal{E}(1)$ and $X \sim \mathcal{G}(2,2)$ ‡.

1.2.2 Unimodality of R(L)

We know from the previous subsection, that the optimal adjustment coefficient R exist if and only if $L > L_0$. The approach to show, that the adjustement coefficient R is unimodal, is the same as the previous section. First, we must ensure that the first derivative $\frac{\partial R}{\partial L}$ cancels exactly once on $L^\star$. And then, we show that $\frac{\partial^2 R}{\partial L^2}(L^\star) < 0$.

∗. using appendix B.7 and $E[(X-L)_+] = \int_L^{+\infty} \overline{F}_X(x)dx$
†. using appendix B.7 and $E[X \wedge L] = \int_0^L \overline{F}_X(x)dx$
‡. the 2 numerical examples considered in the next section. In the previous section, we have $a_0 = 1/3$.

Part 1

Using the implicit function theorem [*], we get

$$\frac{\partial R}{\partial L}(L) = - \left. \frac{\frac{\partial h}{\partial L}(r, L)}{\frac{\partial h}{\partial r}(r, L)} \right|_{r=R}. \tag{1.9}$$

Using the same arguments as the previous section, we have that $\frac{\partial R}{\partial L}(L) = 0$ is equivalent to $\frac{\partial h}{\partial L}(R, L) = 0$. So the optimal retention limit $L^\star$ is such that

$$\frac{\partial h}{\partial L}(R, L^\star) = 0.$$

As shown in appendix A.7, $\frac{\partial X \wedge L}{\partial L} = \mathbf{1}_{(X>L)}$, thus we get

$$\frac{E\left[R(\mathbf{1}_{(X>L^\star)} - C'(L^\star)W)e^{R(X(L^\star)-C(L^\star)W)}\right]}{E\left[e^{R(X(L^\star)-C(L^\star)W)}\right]} = 0,$$

which is equivalent to

$$\underbrace{E\left[(\mathbf{1}_{(X>L^\star)} - C'(L^\star)W)e^{R(X(L^\star)-C(L^\star)W)}\right]}_{f(L^\star)} = 0. \tag{1.10}$$

The equation (1.10) does not always have a unique root. As shown in appendix A.8, the function f defined as the right-hand side of the previous equation has sometimes more than one root, or no roots at all. In the following, we assume now that f has exactly one root $L^\star$ on $]L_0, +\infty[$.

Part 2

The sign of $\frac{\partial^2 R}{\partial L^2}(L^\star)$ can be found when differentiating (1.9). We get

$$\frac{\partial^2 R}{\partial L^2} = - \left. \frac{\frac{\partial^2 h}{\partial L^2}(r, L)}{\frac{\partial h}{\partial r}(r, L)} \right|_{r=R, L=L^\star}.$$

We know that $\frac{\partial h}{\partial r}(R, L) > 0$ from the previous section. So the sign of $\frac{\partial^2 R}{\partial L^2}(L)$ is the same as

$$\frac{\partial^2 h}{\partial L^2}(R, L^\star) = \frac{E\left[R(R(\mathbf{1}_{(X>L^\star)} - C'(L^\star)W)^2 - C''(L)W)e^{R(X(L^\star)-C(L^\star)W)}\right]}{E\left[e^{R(X(L^\star)-C(L^\star)W)}\right]} - \left(\frac{E\left[R(\mathbf{1}_{(X>L^\star)} - C'(L^\star)W)e^{R(X(L^\star)-C(L^\star)W)}\right]}{E\left[e^{R(X(L^\star)-C(L^\star)W)}\right]} \right)^2.$$

As $L^\star$ cancels the equation (1.10), we have

$$\frac{\partial^2 h}{\partial L^2}(R, L^\star) = \frac{E\left[R(R(\mathbf{1}_{(X>L^\star)} - C'(L^\star)W)^2 - C''(L)W)e^{R(X(L^\star)-C(L^\star)W)}\right]}{E\left[e^{R(X(L^\star)-C(L^\star)W)}\right]}.$$

*. cf. appendix A.4

Note that the previous calculi are only valid if X is continuous, in order that the second derivative C'' exists:

$$C''(L) = -(1+\eta_R)\frac{f_X(L)}{E[W]} < 0,$$

and $\frac{\partial \mathbf{1}_{(X>L^\star)}}{\partial L}$ is almost surely null (cf. appendix A.7). Therefore, we obtain

$$\frac{\partial^2 h}{\partial L^2}(R, L^\star) = \frac{E\left[R(R(\mathbf{1}_{(X>L^\star)} - C'(L^\star)W)^2 + (1+\eta_R)\frac{f_X(L)}{E[W]}W)e^{R(X(L^\star)-C(L^\star)W)}\right]}{E\left[e^{R(X(L^\star)-C(L^\star)W)}\right]} > 0,$$

since $P(\mathbf{1}_{(X>L^\star)} = C'(L^\star)W) = 0$ when X and W are continuous. Consequently, we have that the second derivative $\frac{\partial^2 R}{\partial L^2}(L^\star)$ has the opposite sign as $\frac{\partial h}{\partial r}(R, L^\star)$, which is positive as we have already seen. Hence, $\frac{\partial^2 R}{\partial L^2}(L^\star) < 0$, that is to say the function $L \mapsto R(L)$ is unimodal on $]L_0, +\infty[$, when the first part of the sufficient condition is realised.

Conclusion on unimodality

Unlike the proportional case, we are not guarenteed that $L \mapsto R(L)$ is unimodal. However the unimodality is ensured if the equation

$$E\left[(\mathbf{1}_{(X>L)} - C'(L)W)e^{R(X(L)-C(L)W)}\right] = 0$$

has a unique root $L^\star$. Using the fact $\frac{\partial^2 R}{\partial L^2}(L^\star) < 0$, the function $L \mapsto R(L)$ is unimodal on $]L_0, +\infty[$. Otherwise all the roots $L^\star$ are local maxima.

1.2.3 Using other premium calculation principles

As done in the proportional reinsurance case, we study the adjustment coefficient R with other premium principles. We consider the variance, the standard deviation and the exponential premium principles. Let us study those premiums with excess of loss reinsurance with a retention rate L (as usual by η and η_R the loading coefficients). We suppose, as for the expected value principle, that the density of the claim size X exists.

Variance premium principle

The variance premium principle with excess of loss reinsurance is defined as follows

$$C(L) = \frac{E[X] + \eta Var[X]}{E[W]} - \frac{E[(X-L)_+] + \eta_R Var[(X-L)_+]}{E[W]}$$
$$= \frac{E[X \wedge L]}{E[W]} + \frac{\eta Var[X] - \eta_R Var[(X-L)_+]}{E[W]}.$$

The derivatives of $C(L)$ are

$$C'(L) = \frac{\overline{F}_X(L) + 2\eta_R F_X(L) E[(X-L)_+]}{E[W]},$$

and

$$C''(L) = \frac{-f_X(L) - 2\eta_R \overline{F}_X(L) F_X(L) + 2\eta_R f_X(L) E[(X-L)_+]}{E[W]}.$$

As in the previous section with proportional reinsurance, we need to study the 'g' and 'f' functions, and the sign of the second derivative of the adjustment coefficient. With the variance premium principle, we have

$$g(L) = -\eta Var[X] + \eta_R Var[(X-L)_+] \text{ and } g'(L) = -2\eta_R F_X(L) E[(X-L)_+].$$

Hence g is a strictly decreasing function with $g(0) = (\eta_R - \eta)Var[X] > 0$ and g tends to $-\eta Var[X] < 0$ when L tends to $+\infty$. So there is a unique L_0, such that $\forall L > L_0, g(L) < 0$, i.e. $R(L)$ exists on $]L_0, +\infty[$.

Studying the number of roots of the $f(L)$ is very complicated, where f is defined as In the case of excess of loss reinsurance, f is defined as

$$f(L) = E\left[(\mathbf{1}_{(X>L)} - C'(L)W)e^{R(X(L)-C(L)W)}\right].$$

Let us notice $\lim\limits_{L \longrightarrow +\infty} f(L) = 0$ since both functions $\mathbf{1}_{(X>L)}$ and $C'(L) = \frac{\overline{F}_X(L) + 2\eta_R F_X(L) E[(X-L)_+]}{E[W]}$ tends to null. But the solution $L = +\infty$ is not a solution mathematically and in practice. Because this involves that the insurer takes no reinsurance at all. We have also

$$f(L_0) \stackrel{\triangle}{=} E\left[(\mathbf{1}_{(X>L)} - C'(L)W)e^0\right] = -2\eta_R F_X(L_0) E[(X-L_0)_+] < 0.$$

The study of the derivative of f is the same as when using expected value principle in appendix A.8. We are not ensured that f is an increasing function, and so there is a unique $L^\star \in]L_0, +\infty[$ which nullifies f.

The study of the second derivative of f reveals the same impossibility to know its sign. There may be some case where f is an increasing on $]L_0, +\infty[$, hence there is no root, which means the "optimal" * retention limit $L^\star$ will be $+\infty$. Note in this case, we are ensured the $L \mapsto R(L)$ has a horizontal infinite branch when L tends to $+\infty$. Otherwise, the root of f may be unique †.

Finally, the study of the sign of $R''(L)$ is also problematic. We recall its sign is the opposite sign of

$$\frac{\partial^2 h}{\partial L^2}(R, L^\star) = \frac{E\left[R(R(\mathbf{1}_{(X>L^\star)} - C'(L^\star)W)^2 - C''(L)W)e^{R(X(L^\star)-C(L^\star)W)}\right]}{E\left[e^{R(X(L^\star)-C(L^\star)W)}\right]}. \quad (1.11)$$

However, the second derivative of the premium rate, $C''(L)$, is not always negative. Therefore, we are not ensured that $R(L^\star)$ is a maximum, unlike the case of expected value premiums, since (1.11) may be negative. But, we may reasonably think the term $(\mathbf{1}_{(X>L^\star)} - C'(L^\star)W)^2$ to be greater than $C''(L)W$ in average.

This leads to the conclusion that the unimodality of $R(L)$ is not guarenteed even if the f function cancels once.

*. in the sense of the one, which nullifies the first derivative of $R(L)$

†. cf. at the end of appendix A.5

Standard deviation premium principle

The standard deviation premium principle with excess of loss reinsurance is given by

$$C(L) = \frac{E[X] + \eta\sqrt{Var(X)}}{E[W]} - \frac{E[(X-L)_+] + \eta_R\sqrt{Var[(X-L)_+]}}{E[W]}$$
$$= \frac{E[X \wedge L]}{E[W]} + \frac{\eta\sqrt{Var[X]} - \eta_R\sqrt{Var[(X-L)_+]}}{E[W]}.$$

The derivatives of $C(L)$ are

$$C'(L) = \frac{\overline{F}_X(L)}{E[W]} + \eta_R\frac{E[(X-L)_+]F_X(L)}{E[W]\sqrt{Var[(X-L)_+]}},$$

and

$$C''(L) = -\frac{f_X(L)}{E[W]} + \eta_R\frac{-\overline{F}_X(L)F_X(L) + f_X(L)E[(X-L)_+]}{E[W]\sqrt{Var[(X-L)_+]}} - \eta_R\frac{\left(E[(X-L)_+]F_X(L)\right)^2}{E[W]\left(Var[(X-L)_+]\right)^{3/2}}.$$

The study of 'g' and 'f' function is necessary.

$$g(L) = -\eta\sqrt{Var[X]} + \eta_R\sqrt{Var[(X-L)_+]} \text{ and } g'(L) = -\eta_R\frac{E[(X-L)_+]F_X(L)}{\sqrt{Var[(X-L)_+]}} < 0.$$

Since $g(0) = (\eta_R - \eta)\sqrt{Var[X]} > 0$ and g tends to $-\eta\sqrt{Var[X]} < 0$ when L tends to $+\infty$, g has a unique L_0 such that $R(L)$ is only defined on $]L_0, +\infty[$.

The problem of the number of roots of f still persists with the standard deviation premium [*]. Furthermore, the derivative $C''(L)$ is not always negative, hence (1.11) may be negative. As for the variance principle, the function $L \mapsto R(L)$ is not always unimodal, even if f nullifies once.

Exponential premium principle

The exponential premium principle with excess of loss reinsurance is given by

$$C(L) = \frac{\ln\left(E\left[e^{\eta X}\right]\right)}{\eta E[W]} - \frac{\ln\left(E\left[e^{\eta_R (X-L)_+}\right]\right)}{\eta_R E[W]}.$$

The derivatives of $C(L)$ are

$$C'(L) = \frac{1}{E[W]} + \frac{f_X(L)}{\eta_R E[W] E\left[e^{\eta_R (X-L)_+}\right]},$$

and

$$C''(L) = \frac{f'_X(L) + \eta_R f_X(L)}{\eta_R E[W] E\left[e^{\eta_R (X-L)_+}\right]} + \frac{f_X^2(L)}{\eta_R E[W] E^2\left[e^{\eta_R (X-L)_+}\right]}.$$

∗. cf. at the end of appendix A.5

The function g is given by

$$g(L) = E[X \wedge L] - \frac{\ln\left(E\left[e^{\eta X}\right]\right)}{\eta} + \frac{\ln\left(E\left[e^{\eta_R (X-L)_+}\right]\right)}{\eta_R},$$

and its derivative is

$$g'(L) = -F_X(L) - \frac{f_X(L)}{\eta_R E\left[e^{\eta_R (X-L)_+}\right]} < 0.$$

Since $g(0)$ is positive and g tends to $-\frac{\ln\left(E\left[e^{\eta X}\right]\right)}{\eta E[W]} < 0$ when L tends to $+\infty$, the uniqueness of the real $L_0 > 0$ such that $\forall L > L_0, g(L) < 0$, i.e. $R(L)$ exists on $]L_0, +\infty[$.

Again, the number of roots of f is problematic, as the sign of $R''(L)$ when the first derivative $R'(L)$ nullifies, since $C''(L) \geq 0$. Hence, $L \mapsto R(L)$ is not always unimodal.

In conclusion for all these three other premium principles (i.e. variance, standard deviation and exponential principle), there is no guarantees, that $L \mapsto R(L)$ is unimodal on $]L_0, +\infty[$, even if its first derivative nullifies exactly once. That's the main difference with the expected value principle. The following numerical applications will show various examples or counter-examples of unimodality of the adjustment coefficient $R(L)$ *.

1.3 Modelling dependence through copulas

1.3.1 Numerical applications

For the following numerical applications, we models dependence through copulas. Exactly, we will study the optimal retention parameter $\theta^\star$ (either $a^\star$ or $L^\star$) with three different copulas and four different marginal distributions. The studied copulas, which will structure the dependence of the bivariate process (W, X), are :

– the Frank copula :

$$C^F_\alpha(u, v) = \frac{-1}{\alpha} \ln\left(1 + \frac{(e^{-\alpha u} - 1)(e^{-\alpha v} - 1)}{e^{-\alpha} - 1}\right);$$

– the Clayton copula :

$$C^C_\alpha(u, v) = \left(u^{-\alpha} + v^{-\alpha} - 1\right)^{\frac{-1}{\alpha}};$$

– the Gaussian copula :

$$C^N_\alpha(u, v) = H_\alpha(\Phi^{-1}(u), \Phi^{-1}(v));$$

where Φ stands for the standard normal distribution function and H_α the distribution function of a Gaussian vector of mean $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and of covariance matrix $\begin{pmatrix} 1 & \alpha \\ \alpha & 1 \end{pmatrix}$.

We recall that if X and W has a dependence through a bivariate copula C, we have $F_{X,W}(x, w) = C(F_X(x), F_W(w))$. More details on copulas can be found in Nelsen (2006). As for the four different cases of marginal distributions, we study

*. cf. graphs 1.17, 1.18.

- $(X_i)_{i\geq 0} \sim \mathcal{E}(1)$ and $(W_i)_{i\geq 0} \sim \mathcal{E}(1)$;
- $(X_i)_{i\geq 0} \sim \mathcal{G}(2,2)$ and $(W_i)_{i\geq 0} \sim \mathcal{E}(1)$;
- $(X_i)_{i\geq 0} \sim \mathcal{G}(2,2)$ and $(W_i)_{i\geq 0} \sim \mathcal{G}(2,2)$;
- $(X_i)_{i\geq 0} \sim \mathcal{E}(1)$ and $(W_i)_{i\geq 0} \sim \mathcal{G}(2,2)$.

Note that we will always put the claim size type in first and the inter-occurence time in second, in the legends of the following graphics (e.g. `exp(1)/exp(1)` is the first case of marginal distributions).

Since there is no explicit expressions of the Lundberg equation (hence the adjustment coefficient), two approximation methods have been used. These two methods are presented in the two following paragraphs. The first approach uses simulations of the bivariate process $(W_i, X_i)_i$ to compute the adjustment coefficient. In the second method, we discretize the joint distribution function in order to compute the mass probability function.

Using simulation

The main idea of this method is to simulate a bivariate process $U_i = (u_{i,1}, u_{i,2})_{i\geq 0}$ which have a particular copula dependence structure. And then, using the inverse function method, we simulate the marginals W and X. The first step has been carried out thanks to the R* package **copula**†, which produces realisations of U_i. As for the second one, quantile functions of the exponential and the gamma distribution, implemented in R by the functions `qexp` and `qgamma`, are used.

Since we have simulated n samples $(w_i, x_i)_i$ of the bivariate process $(W_i, X_i)_i$, we minimize the squarred differences between left hand side and right hand side of equation (1.3) or (1.8). That is to say we minimize the following empirical function to find the R adjustment coefficient :

$$r \mapsto \left(\frac{1}{n} \sum_{i=1}^{n} e^{r(x_i(\theta) - C(\theta) w_i)} - 1 \right)^2 .$$

Then we maximize the adjustment coefficient R with respect to the retention parameter θ. The both optimizations have been achieved with the `optimize` function of R.

Discretization of the joint distribution function

The main objective of this approach is to compute the mass probability function of the joint distribution function of (W, X), which is given by :

$$f_{W,X}(t_i, x_j) \approx \begin{cases} F_{W,X}(t_i, x_j) - F_{W,X}(t_i, x_{j-1}) - F_{W,X}(t_{i-1}, x_j) + F_{W,X}(t_{i-1}, x_{j-1}) & \text{if } i, j \geq 1 \\ F_{W,X}(t_i, x_j) - F_{W,X}(t_i, x_{j-1}) & \text{if } i = 0 \\ F_{W,X}(t_i, x_j) - F_{W,X}(t_{i-1}, x_j) & \text{if } j = 0 \\ F_{W,X}(0, 0) & \text{if } i, j = 0 \end{cases},$$

where the points $(t_i, x_j)_{0\leq i\leq n_W}^{0\leq j\leq n_X}$ is the grid of discretization with $n_X + 1$ points of space and $n_W + 1$ points of "times". We also use the R package **copula**‡ package to compute the joint distribution

*. the statistical software R (2007)

†. Yan & Kojadinovic (2007)

‡. we rely on the quality of the R package **copula** of Yan & Kojadinovic (2007)

function $F_{W,X}$ for the different copulas, using $F_{W,X}(w,x) = C_\alpha\left(F_W(w), F_X(x)\right)$. As we have the mass probability function, we again minimize the squarred difference of equation (1.3) or (1.8) to find the R adjustment coefficient i.e. the following empirical function

$$r \mapsto \left(\sum_{i=0}^{n_W}\sum_{j=0}^{n_X} e^{r(x_j(\theta)-C(\theta)t_i)} f_{W,X}(t_i, x_j) - 1\right)^2.$$

Subsequently, we maximize the adjustment coefficient R with respect to the retention limit θ.

Note that if not mentioned otherwise, we suppose the premium to be calculated according to the expected value principle.

1.3.2 Proportional reinsurance

Unimodality

First, let us verify numerically that the function $a \mapsto R(a)$ is unimodal. We plot this function for the three studied copulas. The parameters of copulas are 2.5 for the Clayton, 4.5 for the Frank copula and 0.5 for the Gaussian copula. These parameters have been chosen so as to have a Pearson correlation coefficient between claim sizes and claim arrivals around 0.5 *. For this example and all that will follow, we suppose the risk margin are $\eta = 0.2$, $\eta_R = 0.3$ and $n = 10000$ (simulation number). Futhermore, marginals distribution parameters are such that the expectation is 1.

As expected, the graphs of figure (1.1) shows clearly that the function $a \mapsto R(a)$ is unimodal for all copulas and marginals. These graphs have been computed by the simulation method. We can notice that the hump of the curve $(a, R(a))$ is greater when marginals are `gamma(2,2)/gamma(2,2)` than when the distributions are `exp(1)/exp(1)`.

Impact of the parameter dependence α on the optimal retention rate $a^\star$

The results obtained through simulations are first presented. The different parameters are: $\eta = 0.2$, $\eta_R = 0.3$, $n = 10000$ (simulation number).

*. the Pearson correlation is defined as $\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var[X]Var[Y]}}$, hence it depends on the tails of the distribution of X and Y through standard deviations. Thus the Pearson correlation is (slightly) different between `exp(1)/exp(1)` and `exp(1)/gamma(2,2)`. There is no explicit relation between the α parameter of a copula and the Pearson correlation, that's why we use this approximation.

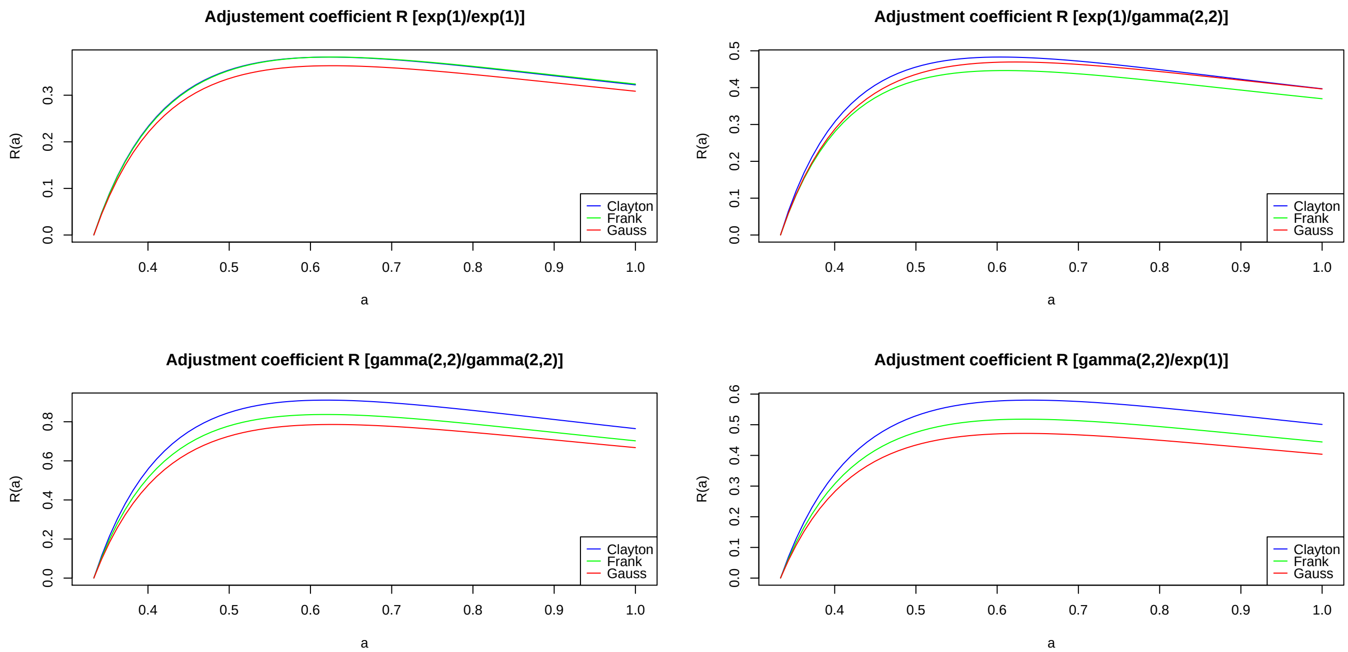

Figure 1.1: Adjustment coefficient

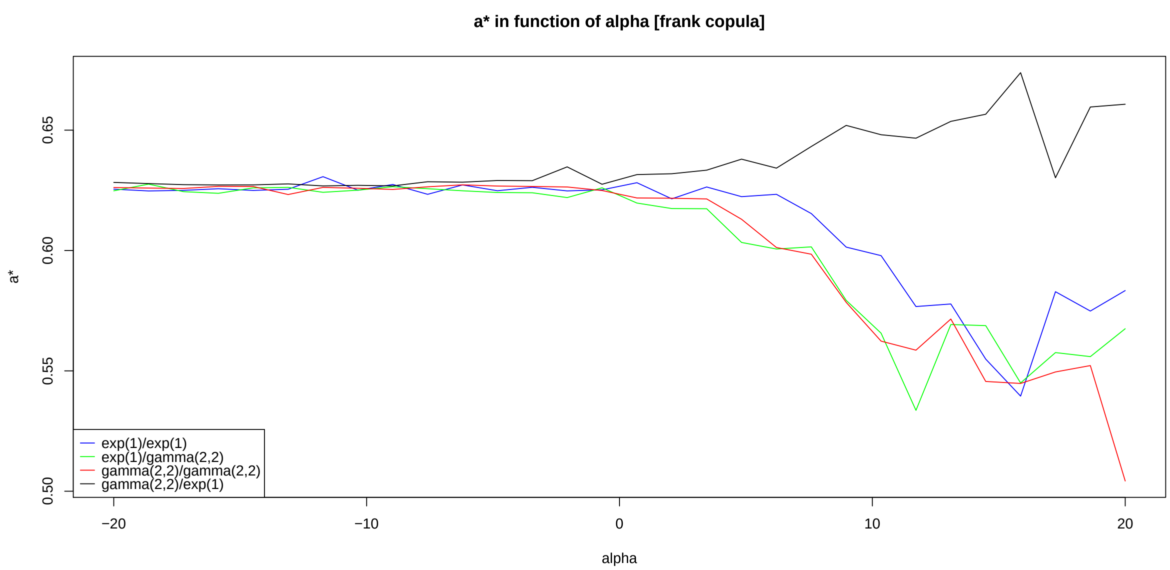

Figure 1.2: Graph of $\alpha \mapsto a^{\star}(\alpha)$ for the Frank copula

In the figures (1.2,1.3), an overall decreasing trend of $a^{\star}$ can be observed for all types of marginals except the `gamma(2,2)/exp(1)` case. When the dependence structure is modelled through a Gaussian copula (fig 1.4), it is quite difficult to see any trends. Moreover, the `gamma(2,2)/exp(1)` marginals case shows that $a^{\star}$ is increasing with α. But it seems that the more the dependence α between inter-occurence times and claim sizes is extreme (i.e. times between two "extreme" claims

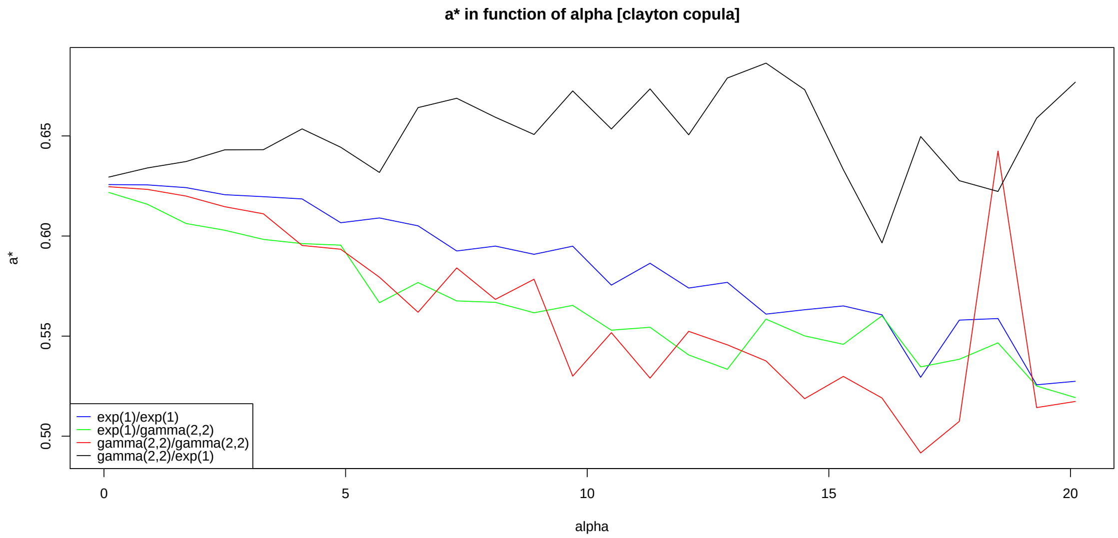

Figure 1.3: Graph of $\alpha \mapsto a^\star(\alpha)$ for the Clayton copula

Figure 1.4: Graph of $\alpha \mapsto a^\star(\alpha)$ for the Gaussian copula

is very long), the more the optimal retention rate $a^\star$ is lower. And so the more, the insurance company "has" (optimally) to reinsure the risk in order to have the maximum safety.

The following results have been carried out by discretization of the joint distribution function. We discretized the claim size X and the claim frequency W on the interval $[0; 20 \times E[X \text{ or } W]]$ or $[0; 10 \times E[X \text{ or } W]]$, whether the distribution of X (or W) is exponential or gamma (respectively). The numbers of points of discretization n_X and n_W are $(125, 125)$ when $X \sim \mathcal{E}(1)$ and $W \sim \mathcal{E}(1)$; $(150, 100)$ when $X \sim \mathcal{G}(2, 2)$ and $W \sim \mathcal{E}(1)$; $(125, 125)$ $X \sim \mathcal{G}(2, 2)$ and $W \sim \mathcal{G}(2, 2)$ and $(100, 150)$ $X \sim \mathcal{E}(1)$ and $W \sim \mathcal{G}(2, 2)$. Hence, the steps of the discretization are respectively $(0.16, 0.16)$, $(0.06, 0.01)$, $(0.08, 0.08)$ and $(0.01, 0.06)$.

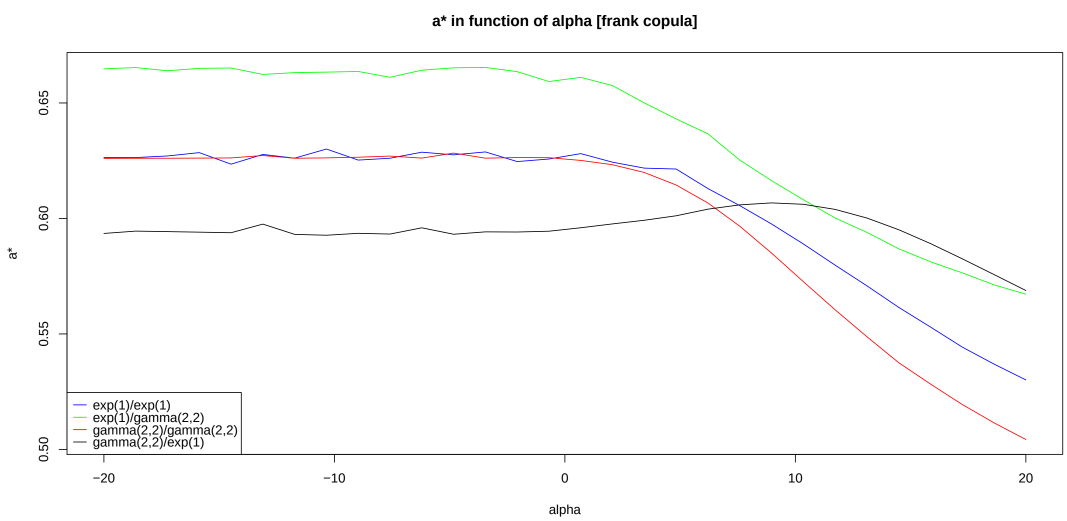

Figure 1.5: Graph of $\alpha \mapsto a^\star(\alpha)$ for the Frank copula

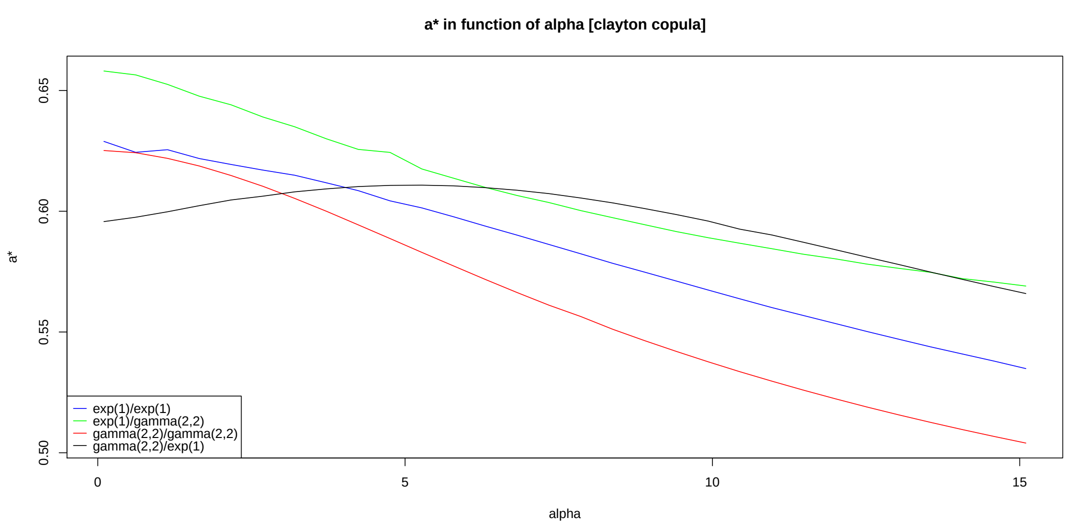

Figure 1.6: Graph of $\alpha \mapsto a^\star(\alpha)$ for Clayton copula

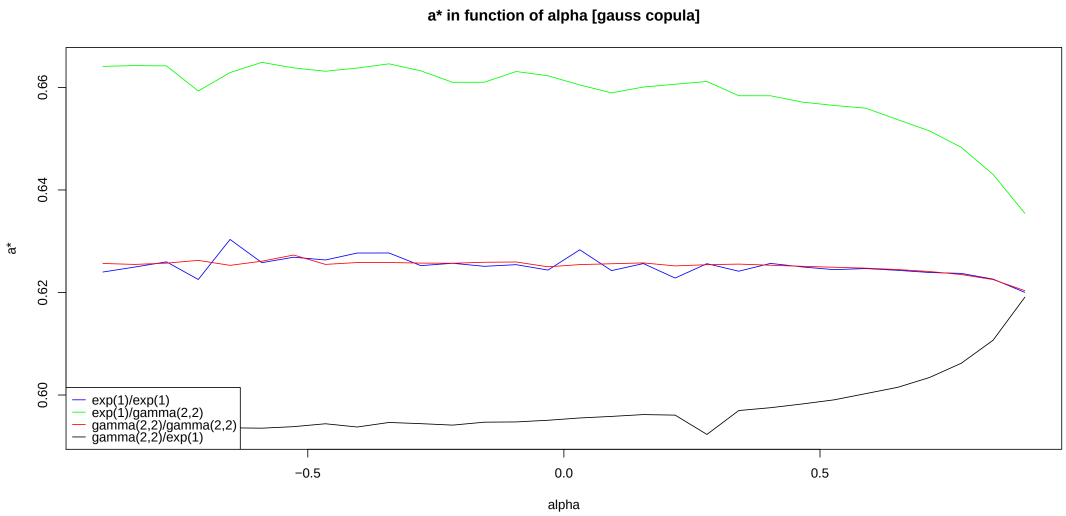

Figure 1.7: Graph of $\alpha \mapsto a^\star(\alpha)$ for the Gaussian copula

The first conclusion of the results, plotted in figures (1.5, 1.6 , 1.7), is the curve $\alpha \mapsto a^\star(\alpha)$ has lost its erratic behavior. Now we can clearly see the decreasing trend of $\alpha \mapsto a^\star(\alpha)$. But for the Archimedian copulas (i.e. Frank and Clayton copulas), the decreasing trend of $\alpha \mapsto a^\star(\alpha)$ is almost incontestable* for a positive dependence ($\alpha > 0$). This phenomena is not so clear for the Gaussian copula, where it seems that $\alpha \mapsto a^\star(\alpha)$ is almost constant.

Moreover, we find the same conclusions on the adjustment coefficient R (not plotted on the previous figures) as Marceau (2007): the adjustment coefficient $R(a^\star)$ is always increasing with the dependence. Behind this, there is the intuitive idea the insurer will have much time to gather a greater amount (when dealing with strong positive dependent risk) of capital if an “extreme” claim raises.

The impact of the premium principles

Until here, we consider in the numerical applications, the expected value premium principle. We know that the unimodality of $R(a)$ still holds when using other premium principles†. The expected value principle does not depend on the tail of the claim size distribution. But for instance, if we use the exponential premium principle, the tail of the claim size distribution is heavily penalized. So with the exponential premium, the optimal retention rate (the abcisse of the maximum of the adjustment coefficient) should be lower than the one when using the expected value principle.

This is shown on the figure (1.8), where the adjustment $R(a)$ coefficient is plotted. As for the standard deviation premium, it is a compromise between the expected value principle (does not depend on the tail of the distribution) and the exponential principle (deeply depends on the tail of

∗. if we exclude the `gamma(2,2)/exp(1)` case.

†. cf. sub section 1.1.3

the distribution). Hence, we expect that the optimal retention with a standard deviation premium principle is between the one of the "exponential premium" case and the one of the "expected value premium" case. That is what we found in the figure (1.9).

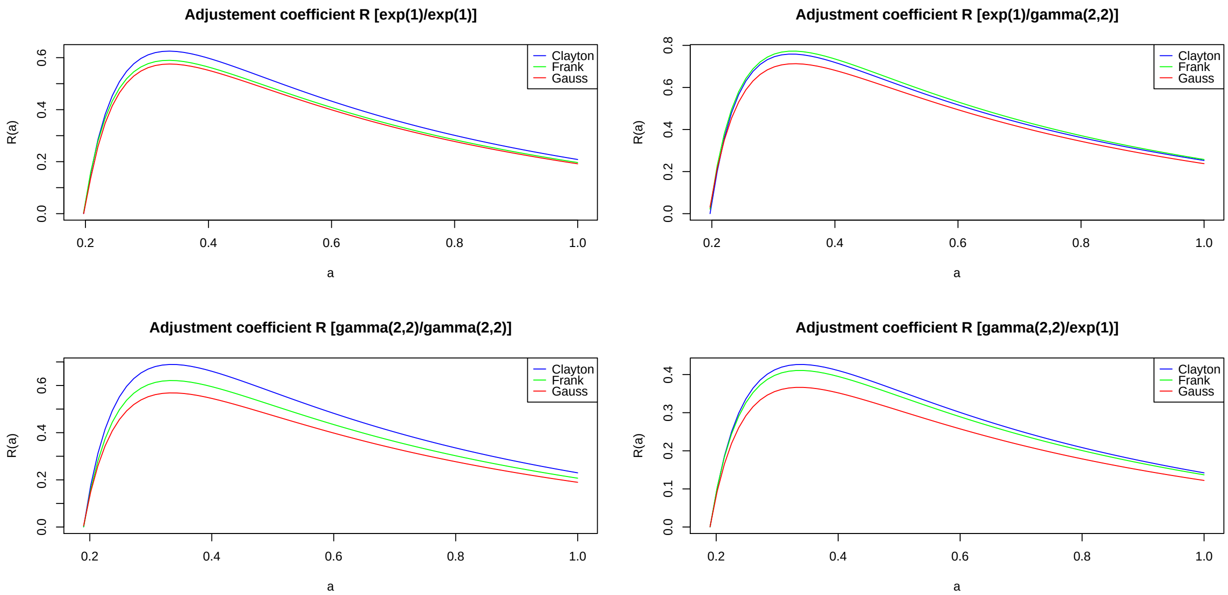

Figure 1.8: Graph of $a \mapsto R(a)$ with the exponential premium principle

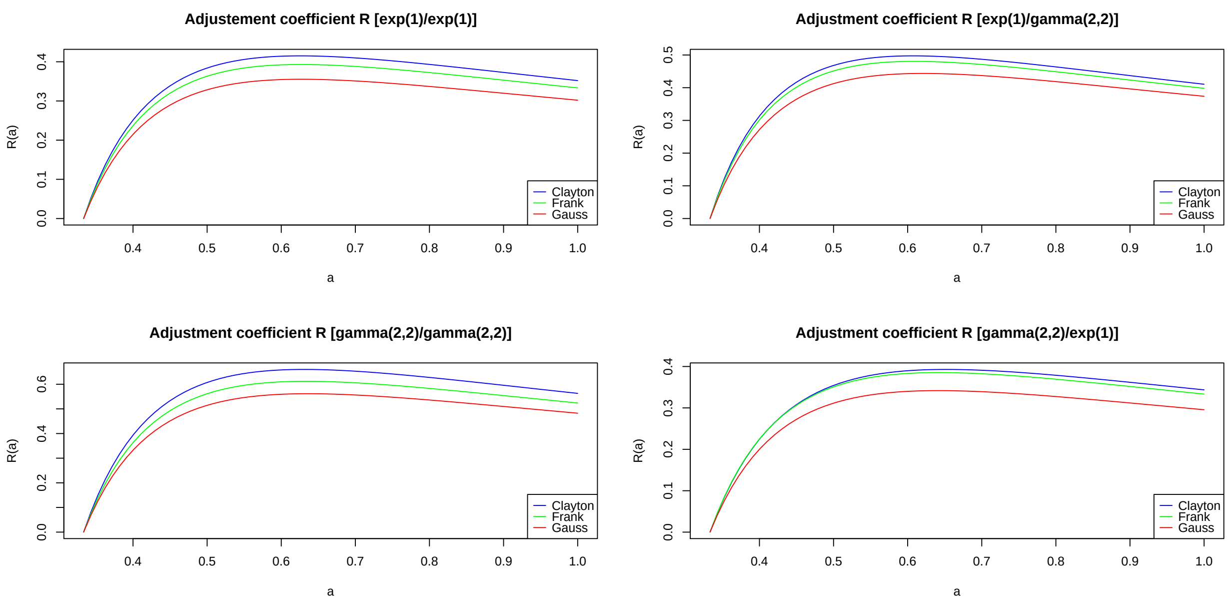

Figure 1.9: Graph of $a \mapsto R(a)$ with the standard deviation premium principle

1.3.3 Excess of loss reinsurance

Unimodality

First, let us see numerically the shape of the function $L \mapsto R(L)$. We plot this function for the three studied copulas. We use the same set of parameters as in the case of proportional reinsurance, that is to say the parameters of copulas are 2.5 for the Clayton, 4.5 for the Frank copula and 0.5 for the Gaussian copula (Pearson correlation around 0.5). We suppose the risk margin are $\eta = 0.2$ and $\eta_R = 0.3$.

As expected, the graphs (1.10) shows that the function $L \mapsto R(L)$ is not always unimodal. With the gamma distribution $\mathcal{G}(100, 100)$ for the claim sizes, the latter function has a local maximum. Hence the excess of loss is not unimodal when this kind of reinsurance is not very appropriate (`gamma(100,100)/gamma(100,100)` has a tiny variance).

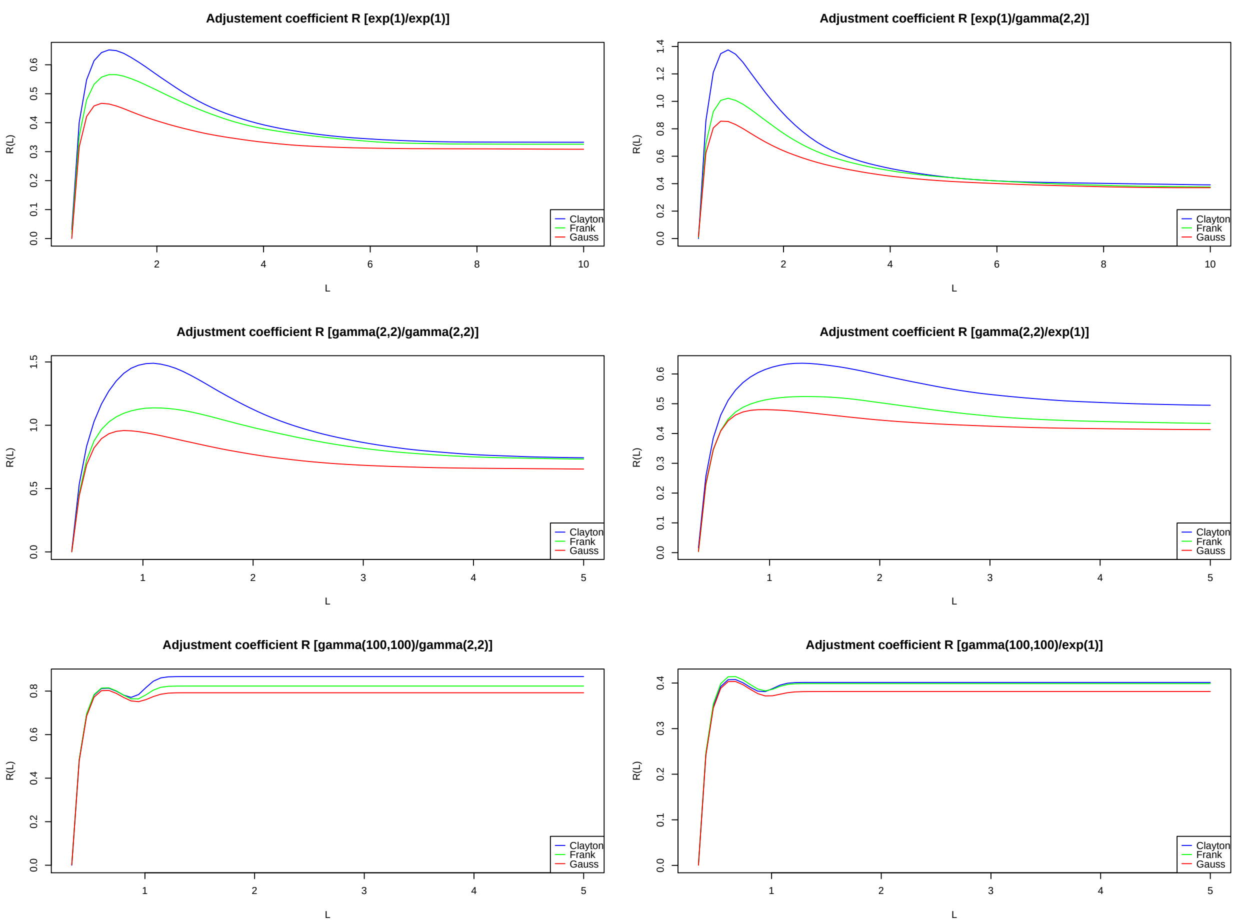

Figure 1.10: Graph of $L \mapsto R(L)$ for different copulas

Impact of the parameter dependence α on optimal retention limit $L^\star$

The results obtained through simulations are first presented. The different parameters are : $\eta = 0.2$, $\eta_R = 0.3$, $n = 10000$ (simulation number). As the previous subsection shows, the studied marginal distributions are in case where the funtion $L \mapsto R(L)$ is unimodal.

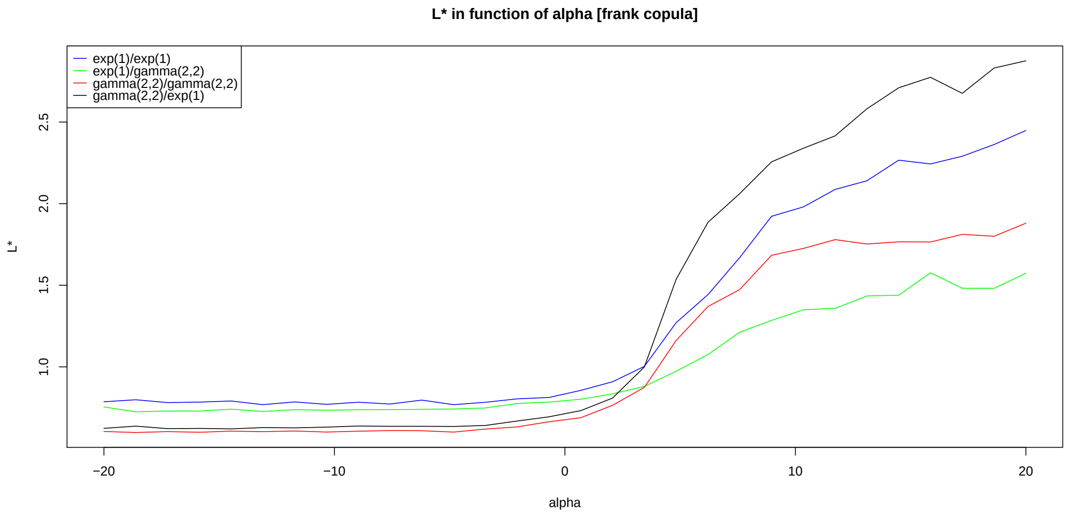

Figure 1.11: Graph of $\alpha \mapsto L^\star(\alpha)$ for the Frank copula

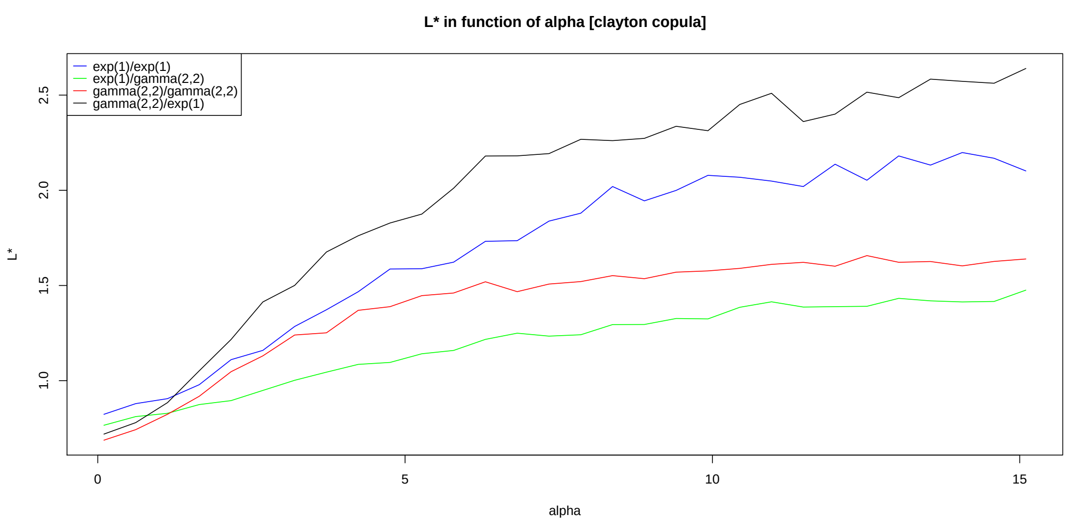

Figure 1.12: Graph of $\alpha \mapsto L^\star(\alpha)$ for the Clayton copula

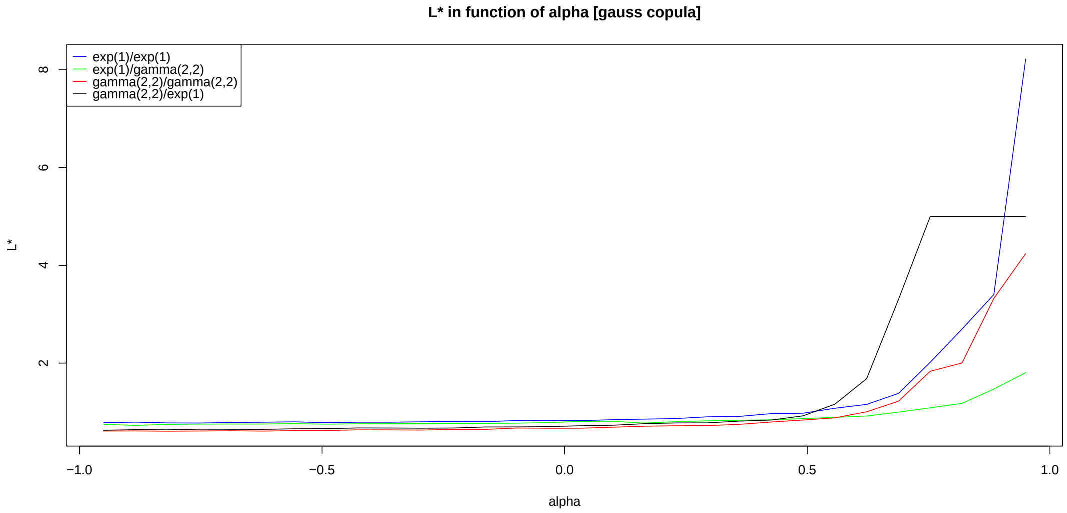

Figure 1.13: Graph of $\alpha \mapsto L^\star(\alpha)$ for the Gaussian copula

In the previous figures (1.11,1.12,1.13), an overall increasing trend of $L^\star$ can be observed for all marginals. In general, the simulation for excess of loss reinsurance gives better results than for proportional reinsurance in terms of smoothness of $L^\star$. As the trend of $L^\star$ is opposite of the $a^\star$ trend, the conclusion for the insurer is the opposite: the more dependence α (between inter-occurence times and claim sizes) is strong, the more the insurer will retain risk (optimally).

The following results have been obtained by discretization with the same grids on claim sizes and claim frequency as the previous subsection.

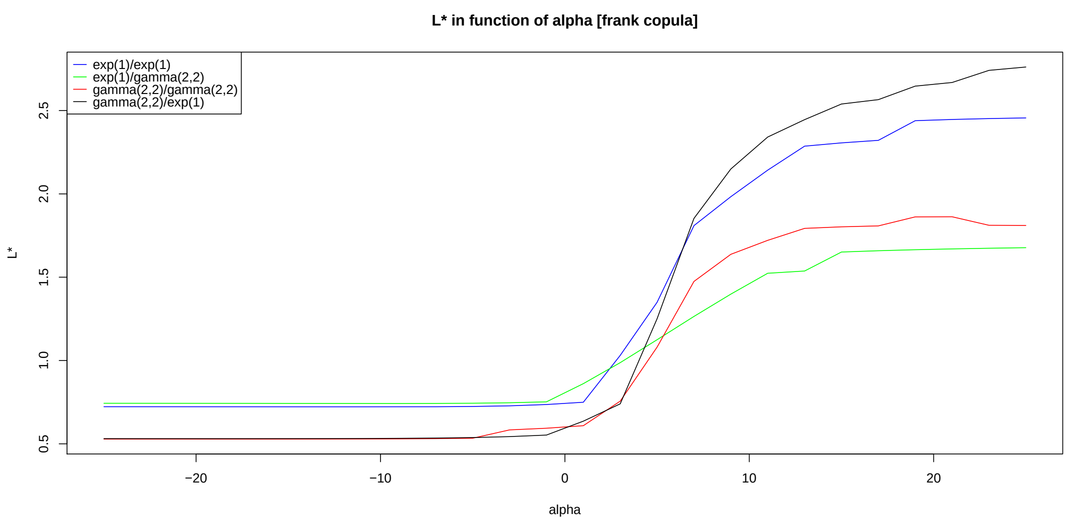

Figure 1.14: Graph of $\alpha \mapsto L^\star(\alpha)$ for the Frank copula

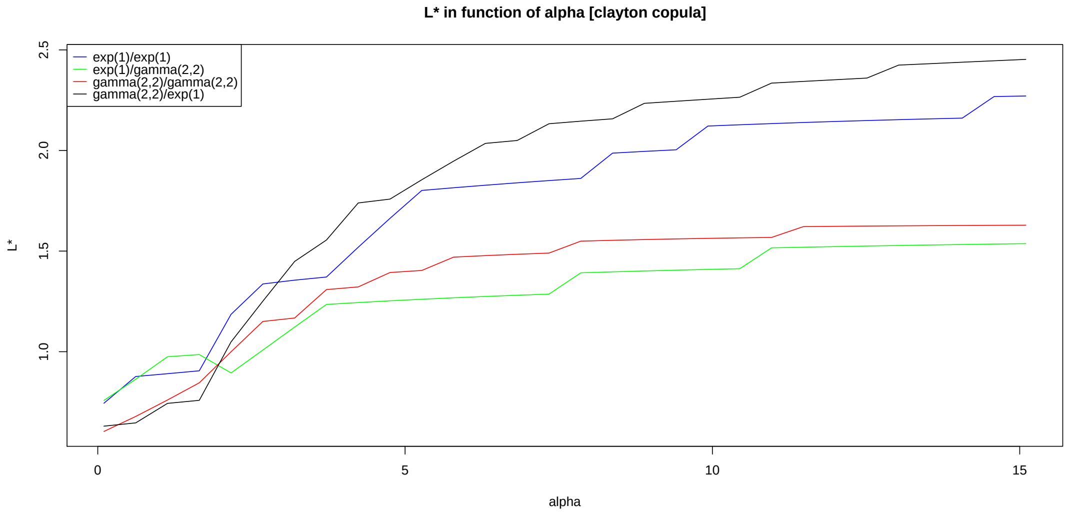

Figure 1.15: Graph of $\alpha \mapsto L^{\star}(\alpha)$ for the Clayton copula

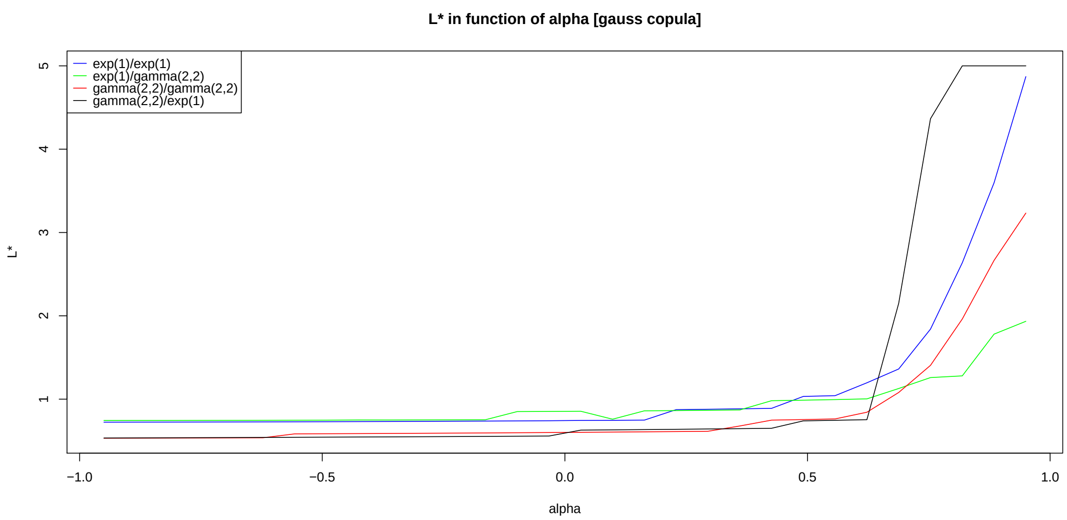

Figure 1.16: Graph of $\alpha \mapsto L^{\star}(\alpha)$ for the Gaussian copula

These results obtained by discretization confirm those by simulation. The function $\alpha \mapsto L^{\star}(\alpha)$ is a increasing function. Let us notice that this function is almost constant for negative dependence and sheerly increasing for positive dependence. It seems also that the optimal limit $L^{\star}$ is bounded in the case of the Frank copula.

Impact of the premium principle

As we did for proportional reinsurance, we analyse the consequencs of the premium principle. We know that the unimodality of $R(L)$ is not always ensured when using other premium principles *. The standard deviation and the exponential principles have the good quality to depend on the tail of the distribution of claim size X. So we expect the optimal retention limit (when it is unique) to be greater than in the case of the expected value principle.

The following graphs show examples or counter-examples of unimodality of $R(L)$. For instance, the exponential principle, plotted in figure (1.17), is unimodal with the Clayton copula (for a Pearson correlation around 0.5). But the Gaussian copula is a counter-example with marginal `gamma(2,2)/gamma(2,2)`. With the standard deviation principle (graph (1.18)), the adjusment coefficient $R(L)$ is not unimodal. However, there is a maximum, which is not unique. The graph (1.19) of the adjustment coefficient $R(L)$ is given in comparison.

If we consider the optimal retention limit as the minimum of retention limits maximizing the adjustment coefficient, let us notice that the optimal retention limits $L^\star$ for the three premium principles are not ordered in the same way as dealing with proportional reinsurance. Actually, with the standard deviation principle, we have the highest optimal retention limit $L^\star$.

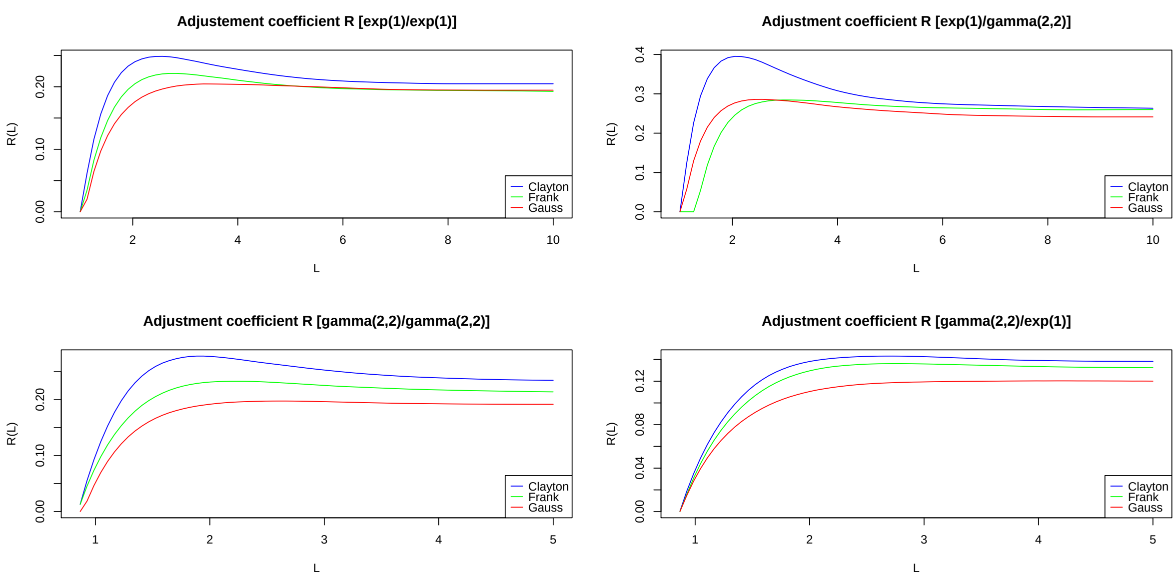

Figure 1.17: Graph of $L \mapsto R(L)$ with an exponential premium principle

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$*$. cf. sub section 1.2.3

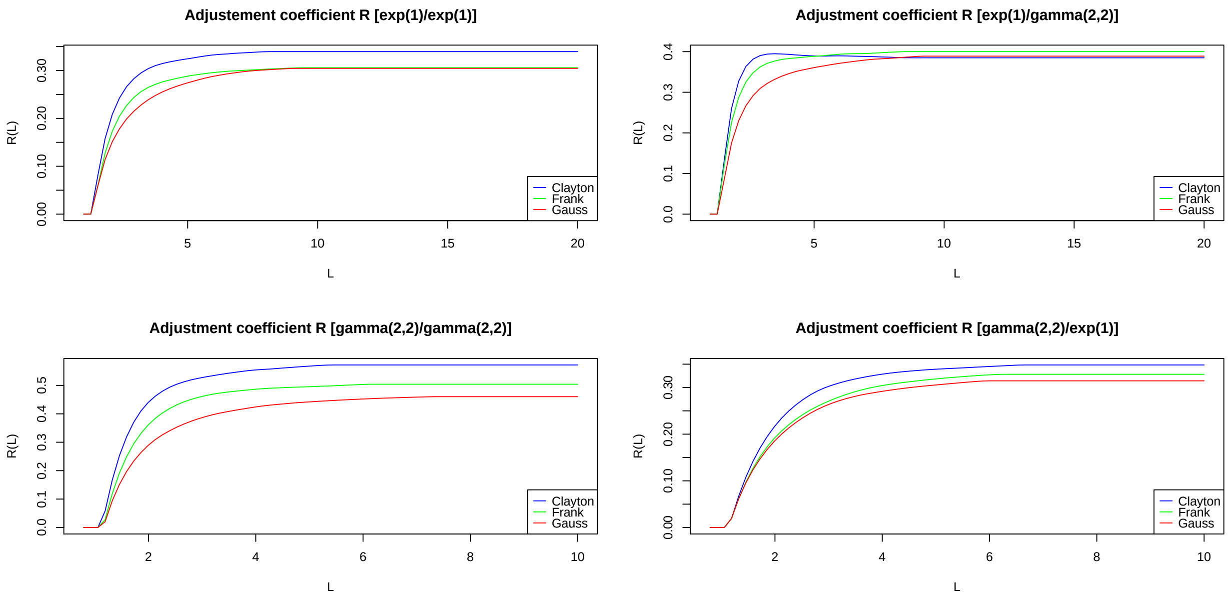

Figure 1.18: Graph of $L \mapsto R(L)$ with a standard deviation premium principle

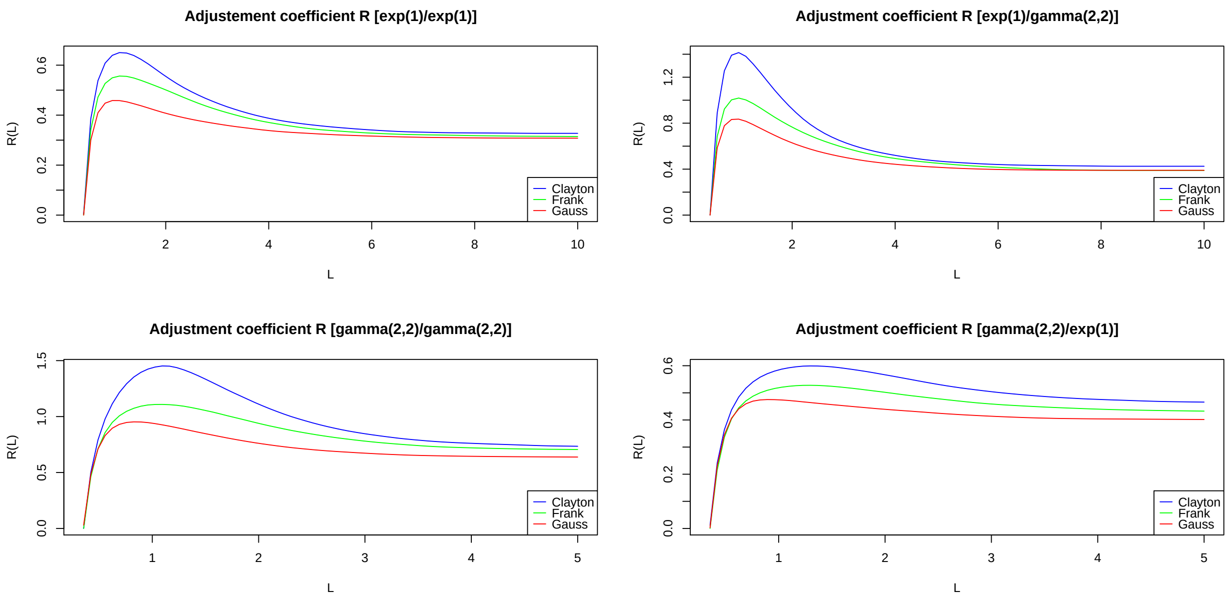

Figure 1.19: Graph of $L \mapsto R(L)$ with an expected value premium principle

1.4 Special cases of comonotonic, independence and countermonotonic copula in proportional reinsurance

In this section, we will study the three particular cases of dependence : the comonotonic, the independence and the countermonotonic copulas for the bivariate process (W, X). The purpose is to study some very particular cases, where the Lundberg equation can be solved explicitly. We suppose the insurer takes proportional reinsurance with the retention rate a. We recall the expression of the three studied copulas :

- Independence copula: $C^{\perp}(u, v) = uv$;
- Comonotonic copula (strongest positive dependence): $C^{+}(u, v) = \min(u, v)$;
- Countermonotonic copula (strongest negative dependence): $C^{-}(u, v) = \max(u + v - 1, 0)$.

1.4.1 Independence copula

We suppose that the inter-occurence times W and the claim sizes X are independent. So the adjustment coefficient R is the positive root of the following equation :

$$M_X(ar)M_W(-rC(a)) = 1, \tag{1.12}$$

where M_X and M_W denote respectively the moment generating function of X and W (if they exist). In section 1.1, we have seen that R is positive if and only if $a > a_0$ (with $a_0 = \frac{\eta_R - \eta}{\eta_R}$). In the following developments, we suppose this situation. From the section 1.1, we also know that the annual premium rate $C(a)$ is a linear function of a :

$$C(a) = \underbrace{\frac{E[X]}{E[W]}(\eta - \eta_R)}_{\alpha} + a \underbrace{(1 + \eta_R)\frac{E[X]}{E[W]}}_{\beta}.$$

If we assume claim size and waiting times are exponentially distributed with parameter λ and δ respectively, then the equation (1.12) becomes

$$\frac{\lambda}{\lambda - ar}\frac{\delta}{\delta + rC(a)} = 1.$$

In this particular case, the adjustment coefficient has an explicit form:

$$R(a) \;=\; \frac{\lambda}{a} - \frac{\delta}{C(a)}.$$

The function $a \mapsto R(a)$ is a unimodal and differentiable function for $a \in]a_0, 1]$, then the derivative R' is $R'(a) = \frac{-\lambda}{a^2} + \frac{\delta C'(a)}{C^2(a)}$. The maximum $R^\star$ satisfies the condition

$$\frac{-\lambda}{a^2} + \frac{\delta C'(a)}{C^2(a)} = 0 \;\Leftrightarrow\; \frac{\lambda}{a^2} = \frac{\delta\beta}{(\alpha + \beta a)^2}.$$

Thus we need to solve the following second order equation

$$a^2(\delta\beta - \lambda\beta^2) - 2\lambda\alpha\beta a - \lambda\alpha^2 = 0.$$

At last, the optimal retention rate verifies

$$a^2((1+\eta_R)\eta_R) + 2a(1+\eta_R)(\eta-\eta_R) + (\eta-\eta_R)^2 = 0,$$

it come the retention rate which minimises ruin probability is

$$a^\star = \frac{\eta_R - \eta}{\eta_R(1+\eta_R)}\left[1+\eta_R+\sqrt{1+\eta_R}\right].$$

Let us notice that the optimal retention rate is independent of the parameters of the exponential distributions. We did numerical applications with $\eta = 0.2$ and $\eta_R = 0.3$, and we find $a^* = 0.625686$.

If we assume that the claim size and claim frequency distributions are gamma with parameter (α, λ) and (α, δ) respectively, then the equation (1.12) becomes

$$\left(\frac{\lambda}{\lambda - ar}\right)^\alpha \left(\frac{\delta}{\delta + rC(a)}\right)^\alpha = 1 \Leftrightarrow \frac{\lambda}{\lambda - ar}\frac{\delta}{\delta + rC(a)} = 1.$$

So this yields to the same calculi as the previous subsection, where we found that the optimal retention rate is $a^* = \frac{\eta_R-\eta}{\eta_R(1+\eta_R)}\left[1+\eta_R+\sqrt{1+\eta_R}\right]$.

In conclusion, we have that explicit expressions of R can be obtained when considering proportional reinsurance and particular marginals. The previous calculi coud be done with other premium principles such as variance, exponential or standard deviation principles. In general, the Lundberg equation (1.12) does not have explicit solution. Futhermore, we could have found explicit expressions of the adjustment coefficient in the case of excess of loss reinsurance (using the truncated moment generating function $M_{X\wedge L}$).

1.4.2 Comonotonicity

The dependence structure between X and W is modelled by the comonotonic copula, also called the Fréchet upper bound: $C^+(u,v) = \min(u,v)$. It is well known that $F_W^{-1}(U)$ has the same distribution as W when U is an uniform distribution $\mathcal{U}(0,1)$ *. Thus $F_W(W)$ has an uniform distribution $\mathcal{U}(0,1)$. The comonotoncity can be characterized by saying that X and W are comonotonic if and only if $F_X^{-1}(F_W(W))$ and W are comonotonic.

Furthermore we suppose that X and W have exponential distribution with parameter λ and δ respectively, we get

$$\begin{aligned} E\left[e^{r(aX-C(a)W)}\right] &= E\left[e^{r(aF_X^{-1}(F_W(W))-C(a)W)}\right] \\ &= \int_0^{+\infty} e^{w(\frac{ra\delta}{\lambda}-rC(a))}\delta e^{-\delta w}dw \\ &= \frac{\delta}{\delta + rC(a) - \frac{ra\delta}{\lambda}}. \end{aligned}$$

Hence the equation $E\left[e^{r(aX-C(a)W)}\right] = 1$ yields to

$$\delta + rC(a) - \frac{ra\delta}{\lambda} = \delta \Leftrightarrow C(a) - \frac{a\delta}{\lambda} = 0.$$

*. property used for the inversion method random simulation

Therefore in the case of comonotonicity, when the marginal distribution are exponential, the adjustment coefficient R does not exist. Other marginals such as gamma distribution don't lead to explicit expressions since we need to explicit expressions of the distribution function.

1.4.3 Countermonotonicity

The last of our three special cases is when the dependence between X and W is countermonotonic. The comonotoncity can be characterized by saying that X and W are countermonotonic if and only if $F_X^{-1}(\overline{F}_W(W))$ * and W are countermonotonic.

In the special case in which X and W are exponential marginal distributions with respective parameters λ and δ, we obtain that the adjustment coefficient equation can be derived to be:

$$\begin{aligned} E[e^{r(aX-C(a)W)}] &= E[e^{r(aF_X^{-1}(\overline{F}_W(W))-C(a)W)}] \\ &= \int_0^{\infty} e^{r(aF_X^{-1}(\overline{F}_W(w))-C(a)w)} f_W(w)dw \\ &= \int_0^1 e^{r(aF_X^{-1}(1-u)-C(a)F_W^{-1}(u))}du \\ &= \int_0^1 e^{r(a\frac{-\ln(u)}{\lambda}+C(a)\frac{\ln(1-u)}{\delta})}du \\ &= \int_0^1 u^{\frac{-ra}{\lambda}}(1-u)^{\frac{C(a)r}{\delta}}du \\ &= E[U^{\frac{-ra}{\lambda}}(1-U)^{\frac{C(a)r}{\delta}}] \\ &= \beta(1-\frac{ra}{\lambda}, 1+\frac{C(a)r}{\delta}), \end{aligned}$$

in terms of beta function. We recall that $\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ †. So when the dependence between claim size ($X \sim \mathcal{E}(\lambda)$) and claim frequency ($W \sim \mathcal{E}(\delta)$) is countermonotonic, the Lundberg equation is

$$\beta(1-\frac{ra}{\lambda}, 1+\frac{C(a)r}{\delta}) = 1. \tag{1.13}$$

However, we must notice that the β function is only defined on $\mathbb{R}_+^\star \times \mathbb{R}_+^\star$. Therefore, the equation (1.13) is only valid if $\frac{ra}{\lambda} < 1$. Let us notice that the equation (1.13) is an extension of the equation, that Albrecher and Teugels found in their article Albrecher & Teugels (2006), which was also expressed in terms of the β function (with no reinsurance). Their equation is a special case of (1.13) with $a = 1$, $\lambda = \lambda_1$, $\delta = \lambda_2$ and $r = -\theta$ (they use the Laplace transform).

1.5 Conditional structure of dependence

In the first two sections, we study a model with no particuler dependence between claim size (X) and inter-occurence times (W), whence we derive results on unimodality of R with proportional and

*. $\overline{F}_W$ denotes the survival function of the random variable W.

†. where the gamma function is defined by $\Gamma(x) = \int_0^{+\infty} x^{t-1}e^{tx}dx$.

excess of loss reinsurance. Then in the numerical applications of the section 1.3, the dependence were structured through different copulas. Now, the hypothesis will take effects on the conditional distribution of X knowing W. As the previous model, we will study the two cases of reinsurance: proportional and excess of loss reinsurance.

1.5.1 Hypothesis

For a fixed $\alpha > 0$, we suppose that the conditional distribution of claim size knowing the claim frequency distribution is as follows

$$F_X^{W=t}(x) = e^{-\alpha t} F_{Y_1}(x) + (1 - e^{-\alpha t}) F_{Y_2}(x), \tag{1.14}$$

where Y_1 and Y_2 are independent positive random variables and independent of W. This model has been studied in Boudreault et al. (2006), where the authors focus on the Gerber-Shiu expected discounted penalty* function (without reinsurance). From this assumption, useful properties can be derived:

- $F_X(x) = M_W(-\alpha) F_{Y_1}(x) + (1 - M_W(-\alpha)) F_{Y_2}(x)$,
- $M_X^{W=t}(x) = e^{-\alpha t} M_{Y_1}(x) + (1 - e^{-\alpha t}) M_{Y_2}(x)$,
- $M_X(x) = M_W(-\alpha) M_{Y_1}(x) + (1 - M_W(-\alpha)) M_{Y_2}(x)$,
- $E[X^n] = M_W(-\alpha) E[Y_1^n] + (1 - M_W(-\alpha)) E[Y_2^n]$,

where M stands for the moment generating function (if it exists).

1.5.2 Proportional reinsurance

Adjustment coefficient equation

In this subsection, we suppose the insurer takes proportional reinsurance with a retention rate $a \in [0, 1]$. The adjustment coefficient R is the positive root of the equation

$$E[e^{r(aX - C(a)W)}] = 1, \tag{1.15}$$

where $C(a)$ denotes the annual premium rate given a retention rate a. Thanks to the previous properties and if the moment generating functions (of W, Y_1 and Y_2) exist, the equation (1.15) can be expressed as

$$M_{Y_1}(ar) M_W(-rC(a) - \alpha) + M_{Y_2}(ar) \left[M_W(-rC(a)) - M_W(-rC(a) - \alpha) \right] = 1.$$

As we already know that the adjusment coefficient $a \mapsto R(a)$ is unimodal on $]a_0, 1]$ in the case of proportional reinsurance, this expression is useful only to compute numerical applications. In this model, no approximations of the Lundberg equation is needed, but there is still no explicit expression of R†.

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*. cf. next chapter

†. R will be obtained through numerical maximisation.

Numerical applications

For numerical applications, we suppose that Y_1 follows a gamma $\mathcal{G}(2,2)$ distribution and Y_1 a gamma $\mathcal{G}(3,3)$ distribution. For the distribution of W, we choose an exponential distribution $\mathcal{E}(1)$ and a gamma distribution $\mathcal{G}(2,2)$. The α parameter takes four different values: 0 (the well known independent risk model), 0.2, 0.4 and 0.6.

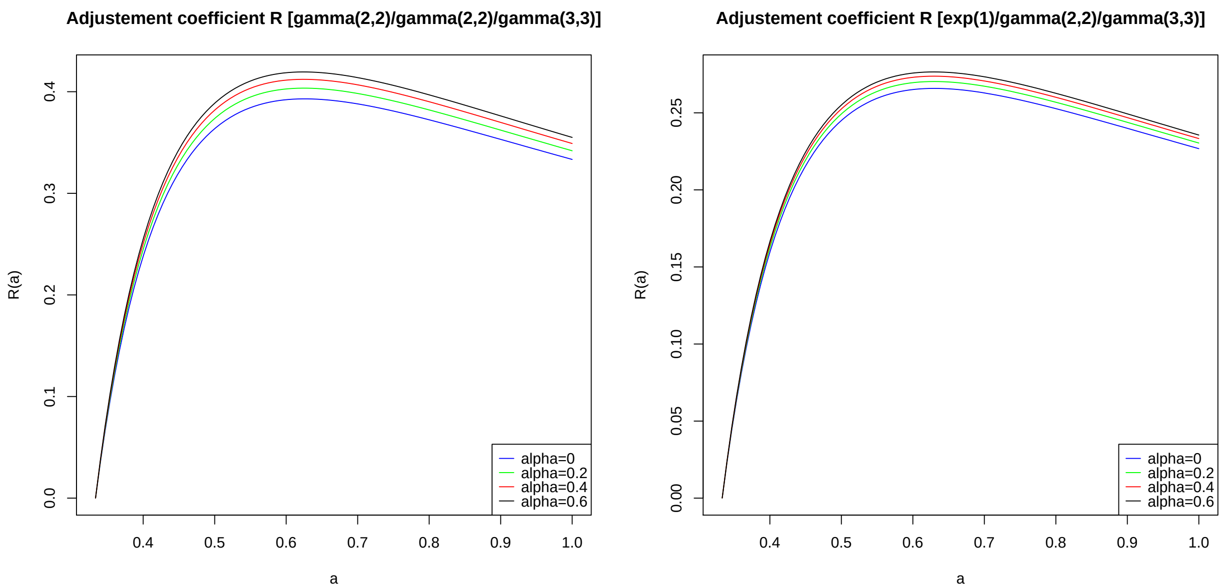

Figure 1.20: Graph of $a \mapsto R(a)$ when $W \sim \mathcal{G}(2,2)$ (left) and $W \sim \mathcal{E}(1)$ (right)

The graphs of figure (1.20) shows that the adjustement coefficient R increases when the parameter α increases. When the parameter α increases, the distribution of Y_2 has a stronger impact on the claim sizes distribution X. So X becomes less risky in terms of variance ($Var[Y_2] = \frac{1}{3}$ vs $Var[Y_1] = \frac{1}{2}$) when α increases. Thus the bigger is α, the greater is the adjustment coefficient R.

Indeed, we can proved that when the parameter α increases, the adjustment coefficient $R(a)$ increases (for all retention rate a), when Y_i follows a gamma distribution $\mathcal{G}(\lambda_i, \lambda_i)$ and W follows a gamma distribution $\mathcal{G}(\chi, \delta)$. The ajdustment coefficient equation can be expressed in the following form

$$M_{Y_2}(ar)M_W(-rC(a)) + M_W(-rC(a) - \alpha)\left[M_{Y_1}(ar) - M_{Y_2}(ar)\right] = 1.$$

Let g be the right hand side of the previous equation. First, we have that $Var(Y_1) > Var(Y_2)$ (i.e. $\lambda_1 < \lambda_2$) implies that $M_{Y_1}(ar) \geq M_{Y_2}(ar)$. Second, the moment generating function of W has an explicit expression:

$$M_W(-rC(a) - \alpha) = \left(\frac{\delta}{\delta + rC(a) + \alpha}\right)^{\chi},$$

which is a decreasing function of α. Thus, for all $a, r > 0$ we have

$$g(\alpha) \stackrel{\triangle}{=} M_{Y_2}(ar)M_W(-rC(a)) + M_W(-rC(a) - \alpha)\left[M_{Y_1}(ar) - M_{Y_2}(ar)\right].$$

is a (strictly) decreasing function of α (for the distribution considered for Y_1, Y_2 and W). That is to say, $\forall a, r > 0,\ \alpha_1 < \alpha_2 \Rightarrow g(\alpha_1) > g(\alpha_2)$, which implies that $R_{\alpha_1} < R_{\alpha_2}$.

Moreover, the impact of the distribution W is as important on the value of R as α parameter (cf. figure (1.20)). When W is exponentially distributed $\mathcal{E}(1)$, the adjustment coefficient R is smaller than when W follows a gamma distribution $\mathcal{G}(2,2)$. In consequence when the variance of W increases, the adjustment coefficient R decreases (a fortiori the optimal adjustment coefficient). This effect is clearly shown on the figure (1.21), where we plot the function $\delta \mapsto R^\star(\delta)$.

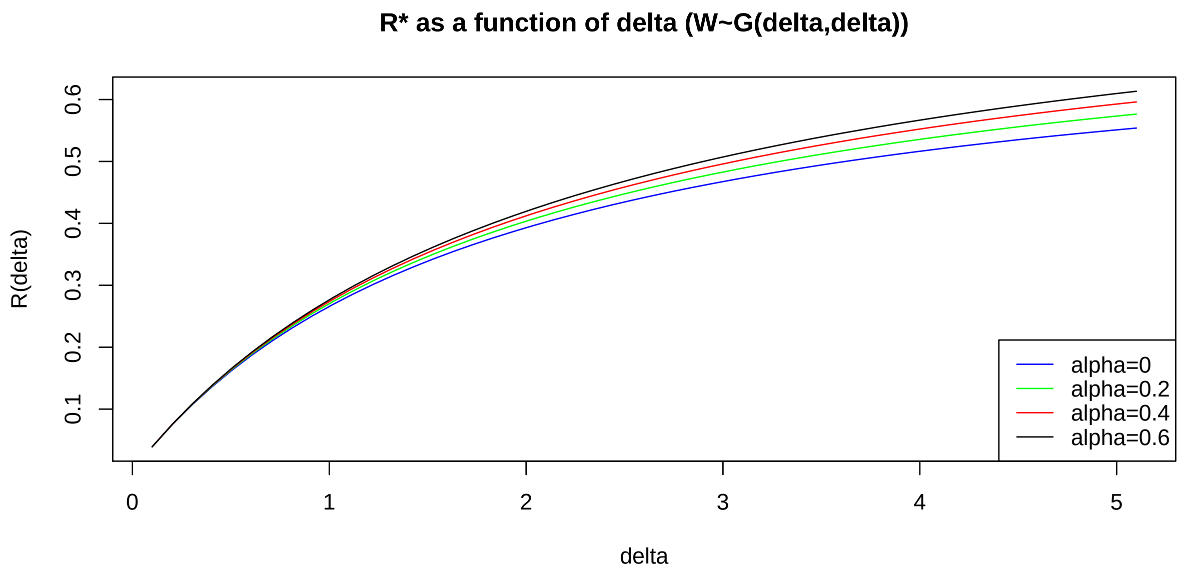

Figure 1.21: Graph of $\delta \mapsto R^\star(\delta)$ when $W \sim \mathcal{G}(\delta, \delta)$

1.5.3 Excess of loss

We assume that the insurer takes excess of loss reinsurance with retention limit L. The adjustment coefficient R is the positive root of the equation

$$E[e^{r(X\wedge L-C(L)W)}]=1, \tag{1.16}$$

where $C(L)$ denotes the annual premium rate given a retention limit L. Again if the moment generating functions exist, the equation (1.16) becomes

$$M_{Y_1\wedge L}(r)M_W(-rC(L)-\alpha)+M_{Y_2\wedge L}(r)\left[M_W(-rC(L))-M_W(-rC(L)-\alpha)\right]=1.$$

Unimodality of R

We have an explicit expression of the limited moment generating functions $M_{Y_i\wedge L}$ when $(Y_i)_{i=1,2}$ follows an Erlang distribution (gamma distribution with integer shape parameter) $\mathcal{G}(n_i,\lambda_i)$

$$M_{Y_i\wedge L}(r)=\left(\frac{\lambda_i}{\lambda_i-r}\right)^{n_i}F_{n_i,\lambda_i-r}(L)+e^{rL}\overline{F}_{n_i,\lambda_i}(L),$$

where $F_{n,\lambda}(L)$ is the distribution function of an Erlang $\mathcal{G}(n,\lambda)$, which is equals to

$$F_{n,\lambda}(x)=1-\sum_{i=0}^{n-1}\frac{(\lambda x)^i}{i!}e^{-\lambda x}.$$

As we proved that the function $L\mapsto R(L)$ is unimodal if and only if the function f has a unique root on $]L_0,+\infty[$ (i.e. the first derivative of $R(L)$ cancels once) *.

Assuming this “conditional” relation of dependence, f is defined by

$$\begin{aligned}
f(L) &\stackrel{\triangle}{=} E\left[(\mathbf{1}_{(X>L)}-C'(L)W)e^{R(X\wedge L-C(L)W)}\right]\\
&= \left[\overline{F}_{Y_1}(L)M_W(-\alpha)+\overline{F}_{Y_2}(L)(1-M_W(-\alpha))\right]e^{RL}M_W(-RC(L))\\
&\quad -C'(L)\left[M_{Y_1\wedge L}(R)M_W(-\alpha)+(1-M_W(-\alpha))M_{Y_2\wedge L}(R)\right]M'_W(-RC(L)),
\end{aligned}$$

when conditioning on W. Assuming Y_i follows an Erlang distribution $\mathcal{G}(n_i,\lambda_i)$, it yields to †

$$\begin{aligned}
f(L)=e^{RL}\overline{F}_X(L)&\left[M_W(-RC(L))-C'(L)M'_W(-RC(L))\right]\\
&-C'(L)M'_W(-RC(L))\left[p_\alpha M_{Y_1}(R)F_{n_1,\lambda_1-R}(L)+(1-p_\alpha)M_{Y_2}(R)F_{n_2,\lambda_2-R}(L)\right],
\end{aligned}$$

where $p_\alpha=M_W(-\alpha))$ and $F_{n,\lambda}(L)$ stands for the distribution function of an Erlang $\mathcal{G}(n,\lambda)$. Just below, we have plotted the f function for the two examples of numerical applications (i.e. $W\sim\mathcal{E}(1)$ and $W\sim\mathcal{G}(2,2)$). The function L has a unique root since it is a combination of continuous strictly convex functions (the moment generating function M_W and $F_{n,\lambda}$).

$*$. cf. sub-section 1.2.2
$\dagger$. cf. appendix A.9

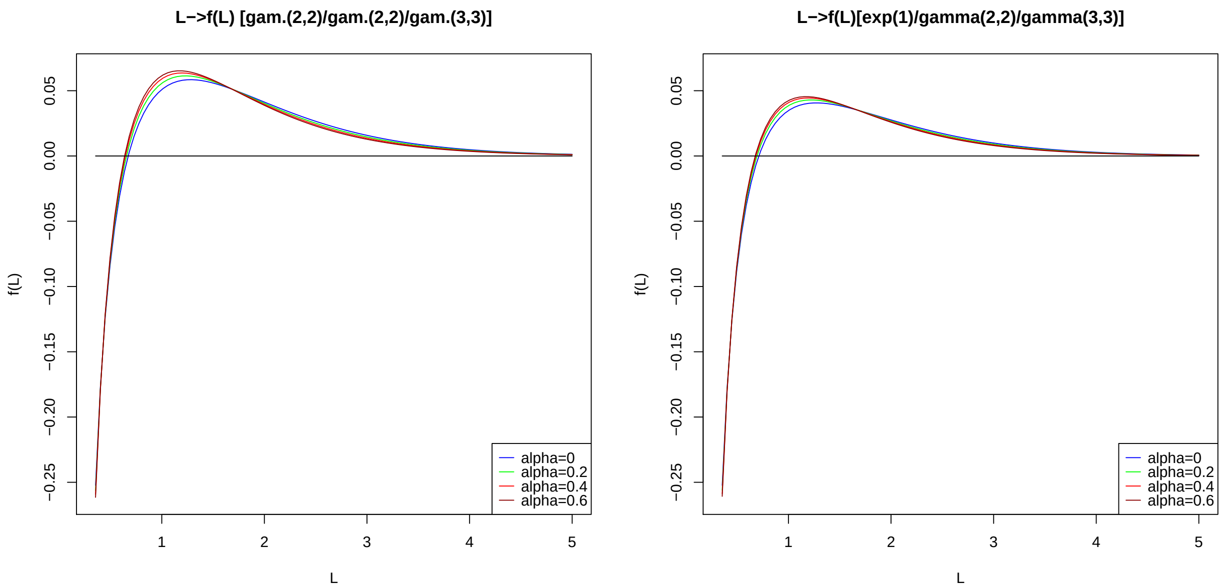

Figure 1.22: Graph of $L \mapsto f(L)$

Numerical applications

For numerical applications, we suppose that Y_1 follows a gamma $\mathcal{G}(2,2)$ distribution and Y_1 a gamma $\mathcal{G}(3,3)$ distribution. For the distribution of W, we choose an exponential distribution $\mathcal{E}(1)$ and a gamma distribution $\mathcal{G}(2,2)$. The parameter takes four different values: 0 (the independent risk model), 0.2, 0.4 and 0.6.

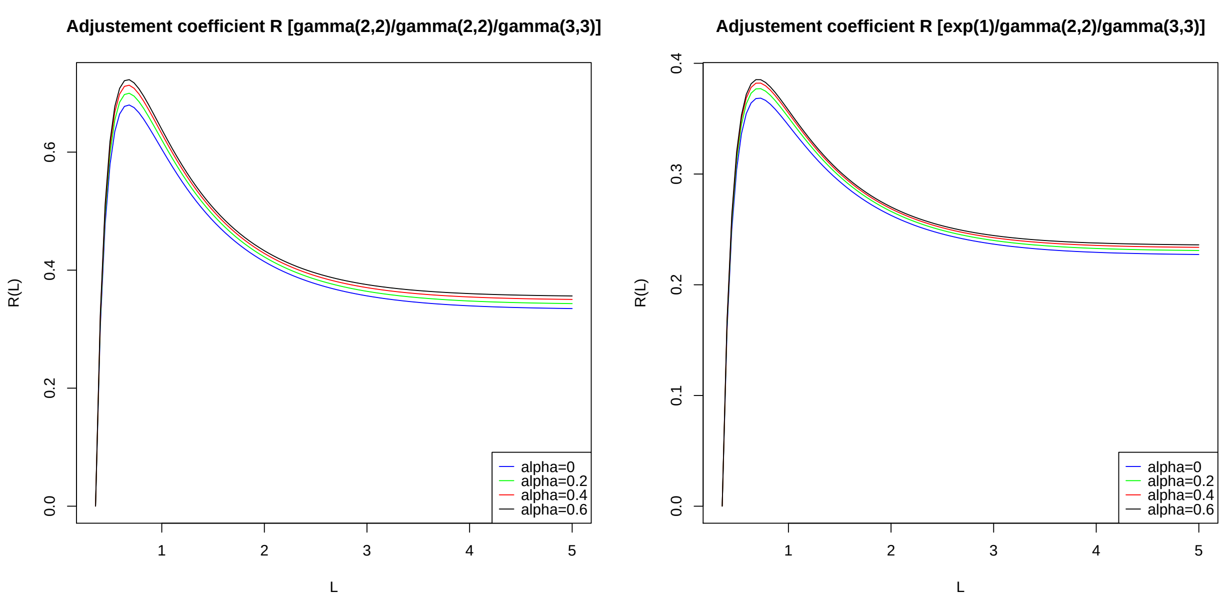

Figure 1.23: Graph of $L \mapsto R(L)$ when $W \sim \mathcal{G}(2,2)$ (left) and $W \sim \mathcal{E}(1)$ (right)

As noticed in the previous sub-subsection (f has a unique root), the function $L \mapsto R(L)$ is unimodal. This is clearly seen on the figure (1.23). Furthermore, the same conclusions as the previous subsection can be drawn from the figure (1.23): the bigger is α, the greater is the adjustment coefficient R; and when the variance of W increases, the adjustment coefficient R decreases. Again we plot the graph of $\delta \mapsto R^\star(\delta)$ when W follows $\mathcal{G}(\delta, \delta)$ (cf. figure (1.24)).

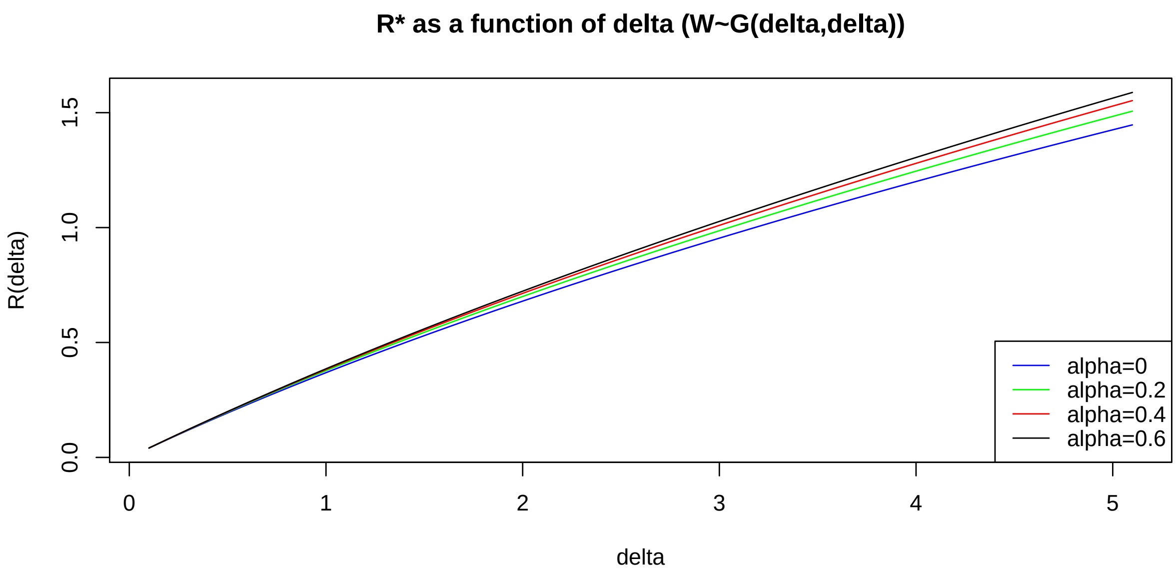

Figure 1.24: Graph of $\delta \mapsto R^\star(\delta)$ when $W \sim G(\delta, \delta)$

1.6 Dependence structure based on common frailty

The main assumption of this approach is that the claim sizes $(X_i)_i$ and the inter-occurence times $(W_i)_i$ knowing the intensity random variable $(\Theta_i)_i$ are conditionnally independent. That is to say we suppose that $(X_i/\Theta_i = \theta, W_i/\Theta_i = \theta)_{i\geq 1}$ is a sequence of independent and identically distributed (i.i.d.) random vectors. We also assumes that $(\Theta_i)_i$ is a sequence of i.i.d. random variables. As Θ_i is assumed to be a discrete distribution on $\{\theta_1, \dots, \theta_m\}$, we have

$$F_{X,W}(x,t) = \sum_{j=1}^{m} P(\Theta = \theta_j) F_X^{\Theta=\theta_j}(x) F_W^{\Theta=\theta_j}(t).$$

Therefore, the adjustment coefficient equation is given by

$$\sum_{j=1}^{m} P(\Theta = \theta_j) M_X^{\Theta=\theta_j}(r) M_W^{\Theta=\theta_j}(-rC) = 1, \tag{1.17}$$

when the moment generating functions exist.

1.6.1 Proportional reinsurance

As in the previous section, we already know, that the adjustment coefficient is unimodal function of the retention rate $a \in]a_0, 1]$ in the case of proportional reinsurance for all kinds of dependence. The goal is to see the influence of the distribution of Θ on the adjustment coefficient. When $X/\Theta = \theta_j$ follows a gamma distribution $\mathcal{G}(\alpha, \theta_j)$ and $W/\Theta = \theta_j$ follows a gamma distribution $\mathcal{G}(\beta, \theta_j)$ the equation (1.17) becomes

$$\sum_{j=1}^{m} P(\Theta = \theta_j) \left(\frac{\theta_j}{\theta_j - ar}\right)^{\alpha} \left(\frac{\theta_j}{\theta_j + rC(a)}\right)^{\beta} = 1.$$

In the numerical applications, the following two distributions of Θ are studied :

1. $P(\Theta = 1) = 0.7$, $P(\Theta = \frac{1}{2}) = 0.2$, $P(\Theta = \frac{1}{3}) = 0.1$, $E[\Theta] = 0.8333$ and $Var(\Theta) = 0.0666$,
2. $P(\Theta = 1) = 0.6$, $P(\Theta = \frac{1}{2}) = 0.25$, $P(\Theta = \frac{1}{4}) = 0.15$, $E[\Theta] = 0.7625$ and $Var(\Theta) = 0.0904$.

These two specific distributions are called in the rest of this paper Θ^1 and Θ^2. We choose these two distributions in order that Θ^2 takes smaller values than Θ^1 (i.e. bigger means for the distributions of X and W).

Furthermore, the claim distributions X and W are
– $X/\Theta = \theta_j \sim \mathcal{E}(\theta_j)$ and $W/\Theta = \theta_j \sim \mathcal{E}(\theta_j)$,
– $X/\Theta = \theta_j \sim \mathcal{G}(2, \theta_j)$ and $W/\Theta = \theta_j \sim \mathcal{E}(\theta_j)$,
– $X/\Theta = \theta_j \sim \mathcal{G}(2, \theta_j)$ and $W/\Theta = \theta_j \sim \mathcal{G}(2, \theta_j)$,
– $X/\Theta = \theta_j \sim \mathcal{E}(\theta_j)$ and $W/\Theta = \theta_j \sim \mathcal{G}(2, \theta_j)$.

First, the figure (1.25) shows that the adjustment coefficient value strongly depends on the distribution of claim sizes and frequency. Let $\overline{e}$ be the ratio of the expectation of X by the expectation of W, i.e. $\overline{e} = \frac{\alpha}{\beta}$. When this ratio increases, the adjusment coefficient $R(a)$ falls dramatically. Especially for the optimal adjusment coefficient $R^\star(a)$, it takes the value around 0.15, 0.11 and 0.07 (0.11, 0.08 and 0.05 respectively) when $\overline{e}$ equals to 0.5, 1 and 2 in the case of Θ^1 (respectively Θ^2).

Second, the distribution of Θ has a big impact on the adjustment coefficient value. When the expectation of Θ decreases (i.e. $E[X]$ and $E[W]$ increases), the adjusment coefficent decreases, especially when $\overline{e} = 0.5$ (i.e. $X/\Theta = \theta_j \sim \mathcal{E}(\theta_j)$ and $W/\Theta = \theta_j \sim \mathcal{G}(2, \theta_j)$).

1.6.2 Excess of loss reinsurance

Now let us study the more interesting case of excess of loss reinsurance. As usual L denotes the retention limit of the insurer. The equation (1.17) becomes

$$\sum_{j=1}^{m} P(\Theta = \theta_j) M_{X \wedge L}^{\Theta=\theta_j}(r) M_{W}^{\Theta=\theta_j}(-rC(L)) = 1. \tag{1.18}$$

In order to prove the unimodality of $R(L)$, we need to show that f has a unique root on $]L_0, +\infty[$ *.

*. cf. sub-section 1.2.2

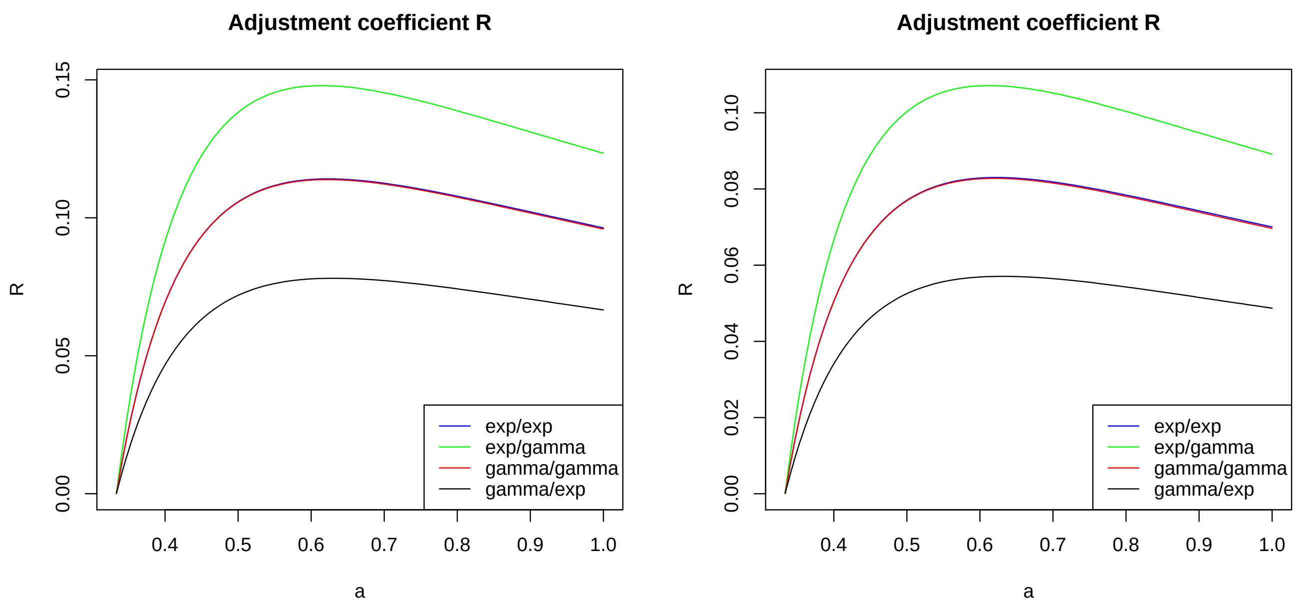

Figure 1.25: Graph of $a \mapsto R(a)$ when $\Theta \sim \Theta^1$ (left) and $\Theta \sim \Theta^2$ (right)

When X and W are exponentially distributed

When we suppose that $X/\Theta = \theta_j \sim \mathcal{E}(\theta_j)$ and $W/\Theta = \theta_j \sim \mathcal{E}(\theta_j)$, the equation (1.18) is given by

$$\sum_{j=1}^{m} P(\Theta = \theta_j) \left[\frac{-\theta_j}{\theta_j - r} + \frac{-re^{-(\theta_j - r)L}}{\theta_j - r} \right] \frac{\theta_j}{\theta_j + rC(L)} = 1.$$

In this case, we have

$$f(L) = \sum_{j=1}^{m} Rp_j M_{W,\theta_j}(-RC(L)) e^{-(\theta_j - R)L} \left[1 + \frac{(1+\eta_R)e^{-\theta_j L}}{(\theta_j + RC(L))(\theta_j - R)} (R - \theta_j e^{(\theta_j - R)L}) \right],$$

where $p_j = P(\Theta = \theta_j)$ and the subscript θ_j denotes the conditional corresponding quantity knowing $\Theta = \theta_j$.

When X and W are gamma distributed

When we suppose that $X/\Theta = \theta_j \sim \mathcal{G}(\alpha, \theta_j)$ and $W/\Theta = \theta_j \sim \mathcal{G}(\beta, \theta_j)$, we have that

$$M_{X \wedge L}^{\Theta = \theta_j}(r) = \left(\frac{\theta_j}{\theta_j - r} \right)^{\alpha} F_{\alpha, \theta_j - r}(L) + e^{rL} \overline{F}_{\alpha, \theta_j}(L),$$

where $F_{\alpha,\theta}$ denotes the distribution function of the gamma distribution $\mathcal{G}(\alpha, \theta)$. Thus the equation (1.18) becomes

$$\sum_{j=1}^{m} P(\Theta = \theta_j) \left[\left(\frac{\theta_j}{\theta_j - r} \right)^{\alpha} F_{\alpha, \theta_j - r}(L) + e^{rL} \overline{F}_{\alpha, \theta_j}(L) \right] \left(\frac{\theta_j}{\theta_j + rC(L)} \right)^{\beta} = 1.$$

In this particular case, the function f is defined as

$$f(L) = \sum_{j=1}^{m} p_j \overline{F}_{\alpha,\theta_j}(L) \left[e^{RL} M_{W,\theta_j}(-RC(L)) - (1+\eta_R) \frac{M_{X\wedge L,\theta_j}(R)}{E[W]} M'_{W,\theta_j}(-RC(L)) \right],$$

with the same notation as above. The function L has a unique root since it is a combination of strictly convex functions (the moment generating function M_{W,θ_j} and $\overline{F}_{\alpha,\theta_j}$). Just below, we have plotted the function f for the two distribution of Θ.

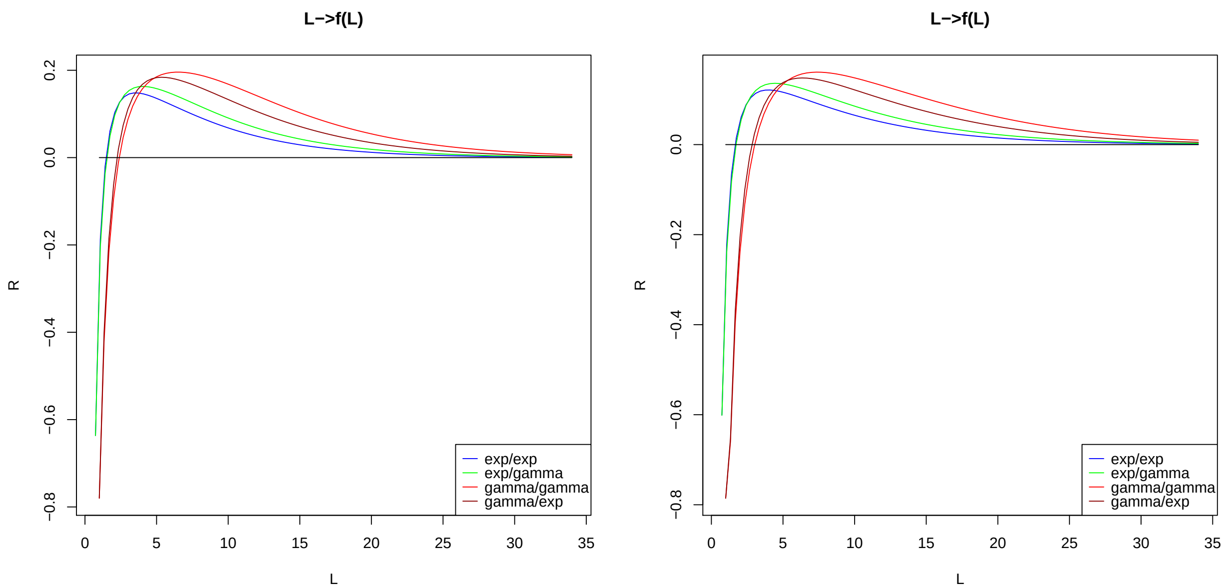

Figure 1.26: Graph of $L \mapsto f(L)$

Numerical applications

The numerical applications, to illustrate the fact that R is a unimodal function of the retention limit, have been carried out with the same parameters as in the case of proportional reinsurance. We consider two examples of the Θ distribution: Θ^1 and Θ^2. As we did throughout the paper, we use the four cases for the distribution of (X, W): `(exp(`θ_j`)/exp(`θ_j`))`, `(exp(`θ_j`)/gamma(2,`θ_j`))`, `(gamma(2,`θ_j`)/gamma(2,`θ_j`))` and `(gamma(2,`θ_j`)/exp(`θ_j`))`.

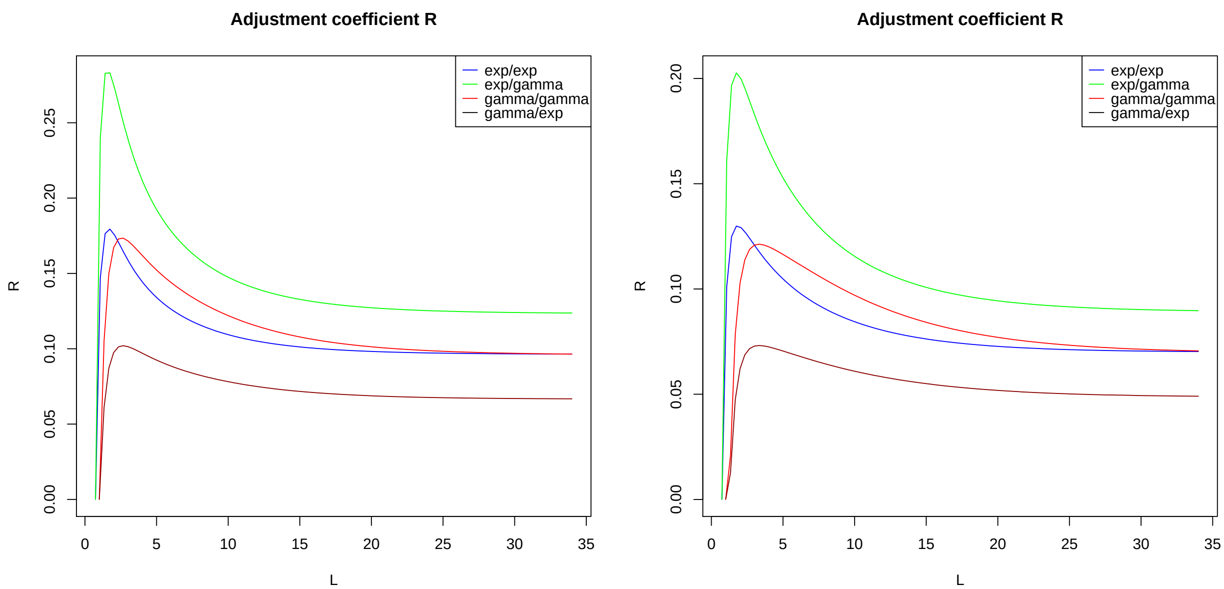

Figure 1.27: Graph of $L \mapsto R(L)$ when $\Theta \sim \Theta^1$ (left) and $\Theta \sim \Theta^2$ (right)

From the figure (1.27), we clealy see that the function $L \mapsto R(L)$ is unimodal. Again we can derive the following conclusions: when the ratio $\overline{e} \triangleq \frac{E[X]}{E[W]} = \frac{\alpha}{\beta}$ increases, the adjusment coefficient $R(a)$ decreases; and when the expectation of Θ decreases, the adjusment coefficent also decreases. The two considered distributions for Θ are such that $E[\Theta^{``1''}] = 0.8333$ and $E[\Theta^{``2''}] = 0.7625$.

1.7 Conclusion

As the purpose of reinsurance is to mititgate the risk of the insurer, the maximisation of $\theta \mapsto R(\theta)$ is an important issue. Whence the question of unimodality of R makes sense, and the question of uniqueness of the optimal retention parameter $\theta^\star$ comes naturally. In section 1.1, we showed that the insurer's adjustment coefficient is a unimodal function of the retention level for proportional reinsurance and all the studied premium principles. But unimodality is not always guarenteed for excess of loss reinsurance, and an assumption on the first derivative of $R(\theta)$ has to be made in section 1.2.

Since we can't find explicit expressions of the adjustment coefficient (so the optimal adjustment coefficient), numerical applications have been carried out through simulation and discretization to illustrate those results. The section 1.4 presented special cases of dependence and claim distribution, which lead to explicit results of the optimal retention rate or the proof of its non-existence. Moreover, the sections 1.5 and 1.6 presents direct applications of sections 1.1 and 1.2 in the two particular models: a conditional structure of dependence and a dependence structure based on common frailty.

Chapter 2

Reinsurance and analysis of ruin measures

Ruin theory is the part of risk theory which focuses on ruin measures. Before Gerber & Shiu (1998), the analysis of ruin measures such as the deficit at ruin, the claim causing the ruin or the ruin probability was not unified. It requires special analysis for all of them. Then Gerber & Shiu (1998) introduced the expected discounted penalty function, whose original goal was to answer two ruin theory problems at the same time: the deficit at ruin and the time of ruin. The analysis of the so-called Gerber-Shiu function let us also to derive some explicit and asymptotic results on the ruin probabilities, the surplus prior to ruin, etc...

In this chapter, we study the Gerber-Shiu function in the Cramér-Lundberg model, when we introduce proportional reinsurance. Unlike the previous chapter, we work with the assumption of independence between claim severity and claim frequency. The aim is to study the influence of reinsurance on ruin measures.

This chapter is structured as follows: in the first section, we summarize the main results of Gerber & Shiu (1998), secondly we introduce reinsurance into the surplus process. Then, numerical applications will be carried out to illustrate the impact of reinsurance on the surplus prior to ruin and the deficit at ruin. Finally, we will conclude.

2.1 The Gerber-Shiu function in the Cramér-Lundberg model

The Cramér-Lundberg model is also referred as the classical risk model in the literature. We consider in this section a risk model where $(N_t)_{t\in\mathbb{R}^+}$, the process of number of claims, follows a Poisson process of parameter λ (i.e. the claim intervals W_i are i.i.d.[*] according to an exponential distribution $\mathcal{E}(\lambda)$) and $(X_i)_{i\in\mathbb{N}^\star}$, the sequence of claim sizes, are i.i.d. positive random variable according to a "generic" random variable X. We assume the independence between the inter-occurence times $(W_i)_i$ and the claim sizes $(X_i)_i$.

$*$. independent and identically distributed

Then we define the ruin time of the insurance company as the first time where the insurance surplus $(U_t)_t \stackrel{\triangle}{=} (u + Ct - S_t)_t$ is (strictly) negative

$$\tau_u = \inf(t > 0, u + Ct - S_t < 0), \tag{2.1}$$

where C denotes the premium rate, u the initial surplus and S_t is the total claim amount at time t (i.e. $S_t = \sum_{i=1}^{N_t} X_i$). If ruin does not occur, $\tau_u = +\infty$. The infinite time ruin probability $\psi(u)$ is defined by $\psi(u) = P(\tau_u < +\infty)$. The premium rate C must satisfy the following condition, so as to avoid almost surely the ruin: $E[X - CW] < 0$, which is equivalent to

$$C = (1 + \eta)\frac{E[X]}{E[W]},$$

where $\eta > 0$ is the safety loading. Let us notice that this condition implies that the surplus process $(U_t)_t$ has a positive drift.

2.1.1 The definition of the discounted penalty function and its associated renewal equation

The Gerber-Shiu discounted penalty function is defined as

$$\varphi_\delta(u) \stackrel{\triangle}{=} E\left[e^{-\delta\tau} w(U_{\tau^-}, |U_\tau|)\mathbf{1}_{(\tau<+\infty)} / U_0 = u\right]. \tag{2.2}$$

where δ is the force of interest *, U_{τ^-} the surplus prior to ruin, and $|U_\tau|$ the deficit at ruin. Denoting by f the joint density of U_{τ^-}, $|U_\tau|$ and τ knowing that $U_0 = u$ (i.e. $f(x, y, t|u) = f^{U_0=u}_{U_{\tau^-},|U_\tau|,\tau}(x, y, t)$), φ_δ can be written as

$$\begin{aligned}
\varphi_\delta(u) &= \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} e^{-\delta t} w(x, y) f(x, y, t|u)dtdxdy, \\
&= \int_0^{+\infty}\int_0^{+\infty} w(x, y) f(x, y|u)dtdxdy,
\end{aligned}$$

where $f(x, y|u)$ stands for the joint density of U_{τ^-} and $|U_\tau|$ knowing that $U_0 = u$. Let us notice that $\varphi_\delta = \psi$ (the ruin probability), with $w(x, y) = 1$ and $\delta = 0$.

From its definition (2.2), a renewal equation verified for φ_δ can be derived. First, Gerber and Shiu obtain the following functional equation by conditioning on the first claim

$$C\varphi'_\delta(u) = (\delta + \lambda)\varphi_\delta(u) - \lambda\int_0^u \varphi_\delta(u - x) f_X(x)dx - \lambda\omega(u), \tag{2.3}$$

where f_X is the density of the random variable X (claim size) and $\omega(u) = \int_u^{+\infty} w(u, x-u) f_X(x)dx$. Let ρ be a positive real. The functional equation (2.3) becomes

$$C\varphi'_{\delta,\rho}(u) = (\delta + \lambda - C\rho)\varphi_{\delta,\rho}(u) - \lambda\int_0^u \varphi_{\delta,\rho}(u - x)e^{-\rho x} f_X(x)dx - \lambda e^{-\rho u}\omega(u), \tag{2.4}$$

*. but δ can also be seen as the variable of a Laplace transform

where $\varphi_{\delta,\rho}(x) = e^{-\rho u}\varphi_\delta(u)$ and λ is the parameter of the Poisson process $(N_t)_t$. The well-known trick to solve (2.4) is to choose $\rho = \xi_1$, solution of the Lundberg equation

$$\delta + \lambda - C\xi = \lambda\widehat{f_X}(\xi), \tag{2.5}$$

where the roots of (2.5) are $\xi_1 = \rho \geq 0$ and $\xi_2 = -R^* < 0$. Note that these two roots are function of δ, and when $\delta = 0$, we have $\rho = 0$.

Finally, we obtain the following renewal equation by integrating (2.4) with respect to u

$$\varphi_\delta = \varphi_\delta * g + h, \tag{2.6}$$

where $*$ stands for the convolution product, $g(x) = \frac{\lambda}{C}\int_0^{+\infty} e^{-\rho z} f_X(x+z)dz$ and $h(x) = \frac{\lambda}{C}\int_x^{+\infty} e^{-\rho(z-x)}\omega(z)dz$.

2.1.2 Solving the renewal equation with the Laplace transform

Applying the Laplace transform on both sides of the equation (2.6), we can solve (2.6)

$$\begin{aligned}\widehat{\varphi}_\delta(\xi) &= \frac{\widehat{h}(\xi)}{1-\widehat{g}(\xi)} \\ &= \sum_{n=0}^{+\infty} \left(\widehat{g}(\xi)\right)^n \widehat{h}(\xi),\end{aligned}$$

where the Laplace transform of g and h are

$$\widehat{g}(\xi) = \frac{\lambda}{c(\rho-\xi)}(\widehat{f_X}(\xi) - \widehat{f_X}(\rho)) \text{ and } \widehat{h}(\xi) = \frac{\lambda(\widehat{\omega}(\xi) - \widehat{\omega}(\rho))}{c(\rho-\xi)}.$$

So, the Laplace transform of φ_δ is

$$\widehat{\varphi}_\delta(\xi) = \frac{\lambda(\widehat{\omega}(\xi) - \widehat{\omega}(\rho))}{c(\rho-\xi) - \lambda(\widehat{f_X}(\xi) - \widehat{f_X}(\rho))}. \tag{2.7}$$

The last expression can be simplified when we consider the penalty function $w(x,y) = 1$. Indeed, we have in this special case

$$\widehat{h}(\xi) = \frac{1}{c(\rho-\xi)}\left(\frac{\lambda}{\xi}(1-\widehat{f_X}(\xi)) + \frac{\delta}{\rho} - C\right).$$

Hence with $w(x,y) = 1$,

$$\widehat{\varphi}_\delta(\xi) = \frac{\lambda\rho(1-\widehat{f_X}(\xi)) + \xi(\delta - C\rho)}{\xi\rho(\lambda(1-\widehat{f_X}(\xi)) + \delta - c\xi)}, \tag{2.8}$$

from which we can derive the Laplace transform of the ruin probability with $\delta = 0$,

$$\widehat{\psi}(\xi) = \frac{\lambda}{\xi} \times \frac{1 - E[X]\xi - \widehat{f_X}(\xi)}{\lambda(1-\widehat{f_X}(\xi)) - C\xi}.$$

Dickson (1998) emphasized that the functional equation (2.3) can be solved directly through its Laplace transform †.

*. R, called the Lundberg coefficient, plays an important role later

†. cf. appendix B.1

2.1.3 Some explicit results on the joint density of U_{τ^-} and $|U_\tau|$

In this sub-section, we focus on the results on $f(x,y|u)$ and $f(x|u)$, respectively the joint density of U_{τ^-} and $|U_\tau|$, the density of U_{τ^-} with initial surplus $U_0 = u$. We have by definition that $f(x|u) = \int_0^{+\infty} f(x,y|u)dy$, which can be simplified to

$$f(x,y|u) = \frac{f(x|u)f_X(x+y)}{\overline{F}_X(x)}{}^{*}. \tag{2.9}$$

In the special case when $u = 0$, we have

$$f(x,y|0) = \frac{\lambda}{C}e^{-\rho x}f_X(x+y) \;\; \text{and} \;\; f(x|0) = \frac{\lambda}{C}e^{-\rho x}\overline{F}_X(x), \tag{2.10}$$

where $\overline{F}_X$ stands for the survival function of X. With the previous results, we have an explicit result for $\varphi_\delta(0)$ when the penalty function $w(x,y) = 1$,

$$\varphi_\delta(0) = \int_0^{+\infty} f(x|0)dx = 1 - \frac{\delta}{C\rho} \underset{\delta\to 0}{\longrightarrow} \frac{\lambda E[X]}{C} = \psi(0).$$

There is a relation between $f(x|u)$ and $f(x|0)$ (so $f(x,y|u)$ and $f(x,y|0)$) that was first found by Dickson (1992) in the special case where $\delta = 0$. This relation was extended by Gerber & Shiu (1998) for all $\delta \geq 0$, where the ruin probability $\psi_\delta(u)$ is defined by $E\left[e^{-\delta\tau+\rho U_\tau}\mathbf{1}_{(\tau<+\infty)}/U_0 = u\right]$ †.

$$f(x|u) = \begin{cases} f(x|0)\frac{e^{\rho u}-\psi_\delta(u)}{1-\psi_\delta(0)} & \text{if } x > u \geq 0 \\ f(x|0)\frac{e^{\rho x}\psi_\delta(u-x)-\psi_\delta(u)}{1-\psi_\delta(0)} & \text{if } 0 < x \leq u \end{cases},$$

where $f(x|0)$ is given by (2.10). An expression of $f(x,y|u)$ in function of $f(x,y|0)$ can be derived ‡

$$f(x,y|u) = \begin{cases} f(x,y|0)\frac{e^{\rho u}-\psi_\delta(u)}{1-\psi_\delta(0)} & \text{if } x > u \geq 0 \\ f(x,y|0)\frac{e^{\rho x}\psi_\delta(u-x)-\psi_\delta(u)}{1-\psi_\delta(0)} & \text{if } 0 < x \leq u \end{cases},$$

where $f(x,y|0)$ is given by (2.10).

2.1.4 Exponentally distributed claim sizes

Now let us study the case where claim sizes are exponentially distributed $X \sim \mathcal{E}(\beta)$ (i.e. $f_X(x) = \beta e^{-\beta x}$). The Lundberg equation (2.5) becomes

$$C\xi^2 + (C\beta - \delta - \lambda)\xi - \beta\delta = 0, \tag{2.11}$$

∗. cf. equation (2.26) in the next section with $a = 1$.

†. the explanation about this defintion of the ruin probability when $\delta > 0$ will follow in the sub-section 2.1.6 on martingales.

‡. cf. equation (2.26) in the next section with $a = 1$.

which leads to $\rho = \frac{\lambda+\delta-C\beta+\sqrt{(C\beta-\delta-\lambda)^2+4C\beta\delta}}{2C}$. R can also be found from this equation, but it is not particularly useful, since we have an explicit formula of the ruin probability. We have the following results

$$f(x,y|0) = \frac{\lambda\beta}{C} e^{-(\rho+\beta)x-\beta y} \quad \text{and} \quad f(x|0) = \frac{\lambda\beta}{C(\beta+\rho)} e^{-\beta y},$$

where ρ is given just below. Furthermore, when the penalty function $w(x,y) = 1$, we have

$$\varphi_\delta(0) = \frac{\lambda}{C(\beta+\rho)}.$$

In this easy example, it is possible to derive the ruin probability $\psi_0(u)$, by inverting its Laplace transform given in (2.8). Indeed, (2.8) becomes

$$\begin{aligned}
\widehat{\psi}_0(\xi) &= \frac{\lambda}{\xi} \times \frac{\frac{\xi}{\beta+\xi} - \frac{\xi}{\beta}}{\lambda(\frac{\xi}{\beta+\xi}) - C\xi} = \lambda \times \frac{1 - \frac{\beta+\xi}{\beta}}{\lambda\xi - C\xi(\beta+\xi)} \\
&= \frac{-\frac{\xi}{\beta}}{(\lambda - C\beta)\xi - C\xi^2} = \frac{\lambda}{\beta C} \times \frac{1}{C\beta - \lambda + \xi},
\end{aligned}$$

Hence, we find the well-known formula of the ruin probability

$$\psi_0(u) = \frac{\lambda}{\beta C} e^{-\gamma u},$$

where $\gamma = C\beta - \lambda > 0$ because of the positive safety loading constraint $E[X - CW] < 0$. When $\delta > 0$, we can't invert easily the Laplace transform of ψ_δ. But as we will see in the subsection 2.1.6, we have

$$\psi_\delta(u) = \frac{\beta - R}{\beta + \rho} e^{-Ru},$$

with

$$\rho = \frac{\lambda + \delta - C\beta + \sqrt{(C\beta - \delta - \lambda)^2 + 4C\beta\delta}}{2C} \quad \text{and} \quad R = \frac{C\beta - \lambda - \delta + \sqrt{(C\beta - \delta - \lambda)^2 + 4C\beta\delta}}{2C}.$$

At last, Gerber & Shiu (1998) get the expression of $f(x,y|u)$ and $f(x|u)$

$$f(x|u) = \begin{cases} \frac{\lambda}{C(R+\rho)} e^{-(\rho+\beta)x} \left[(\beta+\rho)e^{\rho u} - (\beta - R)e^{-Ru}\right] & \text{if } x > u \geq 0 \\ \frac{\lambda(\beta-R)}{C(R+\rho)} e^{-\beta x} \left[e^{Rx} - e^{-\rho x}\right] e^{-Ru} & \text{if } 0 < x \leq u \end{cases},$$

and

$$f(x,y|u) = \begin{cases} \frac{\beta\lambda}{C(\beta-R)} e^{-\rho x} e^{-\beta(x+y)} \left[(\beta+\rho)e^{\rho u} - (\beta - R)e^{-Ru}\right] & \text{if } x > u \geq 0 \\ \frac{\beta\lambda(\beta-R)}{C(R+\rho)} e^{-\beta(x+y)} \left[e^{Rx} - e^{-\rho x}\right] e^{-Ru} & \text{if } 0 < x \leq u \end{cases}.$$

Let us notice that $f(x|u)$ can be obtained either by integration of $f(x,y|u)$ or using the fact $f(x,y|u) = f(x|u) f_X(y)$ when $X \sim \mathcal{E}(\beta)$ *.

*. cf. (2.9)

2.1.5 Asymptotic results

From the renewal equation (2.6), one can apply the key renewal theorem *. We are in the case of a defective renewal equation, hence $R > 0$. So the equation $\widehat{f}(-R) = 1$ is the Lundberg equation (2.5) and $R = -\xi_2$. Therefore, we have the following asymptotic result for

$$\varphi_\delta(u) \underset{+\infty}{\sim} \frac{\widehat{h}(-R)}{-\left(\widehat{g}\right)'(-R)} e^{-Ru},$$

which is equivalent to

$$\varphi_\delta(u) \underset{+\infty}{\sim} \frac{\lambda \int_0^{+\infty} \int_0^{+\infty} w(x,y)(e^{Rx} - e^{-\rho x}) f_X(x+y) dx dy}{-\lambda \left(\widehat{f}_X\right)'(-R) - C} e^{-Ru}. \tag{2.12}$$

From the previous result (2.12), we can derive asymptotic results for $f(x,y|u)$ and $f(x|u)$

$$f(x,y|u) \underset{+\infty}{\sim} \frac{\lambda(e^{Rx} - e^{-\rho x}) f_X(x+y)}{-\lambda \left(\widehat{f}_X\right)'(-R) - C} e^{-Ru} \quad \text{and} \quad f(x|u) \underset{+\infty}{\sim} \frac{\lambda(e^{Rx} - e^{-\rho x}) \overline{F}_X(x)}{-\lambda \left(\widehat{f}_X\right)'(-R) - C} e^{-Ru},$$

which yields to, when $X \sim \mathcal{E}(\beta)$

$$f(x,y|u) \underset{+\infty}{\sim} \frac{\lambda\beta(e^{Rx} - e^{-\rho x}) e^{-\beta(x+y)}}{-\lambda\beta(\beta - R)^{-2} - C} e^{-Ru} \quad \text{and} \quad f(x|u) \underset{+\infty}{\sim} \frac{\lambda(e^{Rx} - e^{-\rho x}) \overline{F}_X(x)}{-\lambda\beta(\beta - R)^{-2} - C} e^{-Ru}.$$

In the special case where the penalty function $w(x,y) = 1$, the equivalent in $+\infty$ (2.12) becomes

$$\psi_\delta(u) \underset{+\infty}{\sim} \frac{\delta}{-\lambda \left(\widehat{f}_X\right)'(-R) - C} \left(\frac{1}{R} + \frac{1}{\rho}\right) e^{-Ru} \underset{\delta \to 0}{\longrightarrow} \frac{C - \lambda E[X]}{-\lambda \left(\widehat{f}_X\right)'(-R) - C} e^{-Ru} \dagger,$$

which yields to, when $X \sim \mathcal{E}(\beta)$

$$\psi_\delta(u) \underset{+\infty}{\sim} \frac{\delta}{-\lambda\beta(\beta - R)^{-2} - C} \left(\frac{1}{R} + \frac{1}{\rho}\right) e^{-Ru}.$$

2.1.6 Martingales

As we have just seen, the adjustment coefficent R or the Lundberg coefficient plays a key role in the previous sub-section, since it is the constant that makes the equation (2.6) a proper renewal equation. Furthermore, Gerber & Shiu (1998) has provided another interpretation for the adjustment coefficient with martingales ‡.

∗. cf. appendix B.2

†. which is called the Cramér-Lundberg approximation in the literature.

‡. cf. the appendix B.3 for the definition of a martingale in a continuous time

Let us define the process $(V_{\xi,t})_t$ as $\left(e^{-\delta t+\xi U_t}\right)_{t\geq 0}$. Because of the stationary and independent increments of the surplus process $(U_t)_t$ §, $(V_{\xi,t})_t$ is a martingale if and only if

$$E\left[e^{-\delta t+\xi U_t}/U_0=u\right]=e^{\xi u}. \tag{2.13}$$

This equation is equivalent to the Lundberg equation. Indeed, we have that the left-hand side of (2.13) is given by

$$E\left[e^{-\delta t+\xi U_t}/U_0=u\right]=e^{-\delta t+\xi(u+Ct)}E\left[e^{\xi S_t}\right]=e^{-\delta t+\xi(u+Ct)}e^{\lambda t(\widehat{f}_X(\xi)-1)},$$

since $(S_t)_t$ is a compound Poisson process of parameter λ. Hence, (2.13) is

$$-\delta t+\xi(u+Ct)+\lambda t(\widehat{f}_X(\xi)-1)=\xi u,$$

which is the Lundberg equation (2.5). Hence, either ρ or $-R$ makes $(V_{\xi,t})_t$ a martingale. This explains the definition of the ruin probability when $\delta>0$, $E\left[e^{-\delta\tau+\rho U_\tau}\mathbf{1}_{(\tau<+\infty)}/U_0=u\right]$ (i.e. φ_δ when $w(y)=e^{\rho y}$), which is the usual definition $P(\tau<+\infty/U_0=u)$ when $\delta=0$.

As we want to calculate the quantity $E\left[e^{-\delta\tau-RU_\tau}\mathbf{1}_{(\tau<+\infty)}/U_0=u\right]$, one may wonder if the equality (2.13) still holds for the ruin time τ * (a stopping time), when $\xi=\rho$ or $-R$. Hopefully, it holds thanks to the optional sampling theorem †. That is to say

$$e^{-Ru}=E\left[e^{-\delta\tau-RU_\tau}/U_0=u\right]=E\left[e^{-\delta\tau-RU_\tau}\mathbf{1}_{(\tau<+\infty)}/U_0=u\right]+0,$$

thus we have this useful formula

$$E\left[e^{-\delta\tau-RU_\tau}\mathbf{1}_{(\tau<+\infty)}/U_0=u\right]=e^{-Ru},\ \text{with } \delta\geq 0,\ u\geq 0. \tag{2.14}$$

Furthermore, Gerber and Shiu give a probabilistic interpretation of the quantity $e^{-\rho(x-u)}$ in their article Gerber & Shiu (1998). They define the stopping time T_x as $\inf(t>0, U_t=x)$ with $x>U_0=u$ (i.e. the first time the surplus crosses the barrier level x). Again, they apply the optional sampling theorem ‡ on the martingale $(V_{\rho,t})_t$. So we have

$$e^{\rho u}=E\left[e^{-\delta T_x+\rho U_{T_x}}/U_0=u\right]=e^{\rho x}E\left[e^{-\delta T_x}/U_0=u\right],$$

which is equivalent to

$$E\left[e^{-\delta T_x}/U_0=u\right]=e^{-\rho(x-u)}.$$

The last equality is the Laplace transform of the random variable T_x (i.e. Laplace transform of its density). From this result, one can obtain once again the formula of the sub-section 2.1.3, where the quantity $e^{-\rho x}$ appears in almost every equations.

Finally, the authors get the Laplace transform of the finite ruin probability, which is defined as

$$\psi(u,t)=P(\tau<t/U_0=u),$$

§. cf. the appendix B.4
∗. defined by (2.1)
†. cf. the appendix B.5
‡. cf. the appendix B.5

and

$$\widehat{\psi}(\xi,\delta) \triangleq \int_0^{+\infty}\int_0^{+\infty} e^{-\xi u-\delta t}\psi_\delta(u,t)dudt = \frac{1}{\delta\xi}\frac{\lambda\rho(1-\widehat{f_X}(\xi))+\xi(\delta-C\rho)}{\lambda(1-\widehat{f_X}(\xi))-c\xi},$$

and so

$$\widehat{\phi}(\xi,\delta) = \frac{1/\xi-1/\rho}{\lambda(1-\widehat{f_X}(\xi))-c\xi},$$

where $\phi(u,t) \triangleq 1-\psi(u,t)$ stands for the survival probability.

2.1.7 Exponentally distributed claim sizes (Continued)

We already present some results when the claim size X is exponentially distributed $\mathcal{E}(\beta)$. First, we will show the explicit formula of the ruin probability when $\delta \geq 0$. Since $f(x,y|u) = f(x|u)f_X(y)$ *, we have by the relation (2.14)

$$E\left[e^{-\delta\tau}\mathbf{1}_{(\tau<+\infty)}/U_0=u\right] = \frac{\beta-R}{\beta}e^{-Ru}.$$

Thanks to the previous formula and $f(x,y|u) = f(x|u)f_X(y)$, we can obtain explicit formula of φ_δ when $w(x,y) = w(y)$:

$$\varphi_\delta(u) = E\left[e^{-\delta\tau}w(|U_\tau|)\mathbf{1}_{(\tau<+\infty)}/U_0=u\right] = \int_0^{+\infty} w(y)\beta e^{-\beta x}dy \times \frac{\beta-R}{\beta}e^{-Ru}.$$

This yields to

$$\varphi_\delta(u) = \widehat{w}(\beta)(\beta-R)e^{-Ru},$$

from which we can derive the ruin probability ψ_δ when $w(y) = e^{-\rho y}$

$$\psi_\delta(u) = \frac{\beta-R}{\beta+\rho}e^{-Ru}.$$

If we compare this relation to $\psi_0(u) = \frac{\lambda}{\beta C}e^{-\gamma u}$, the relation is not obvious. But since ρ and $-R$ are roots of the Lundberg equation (2.11), which is in this case quadratic, the product $(-R\rho)$ and the sum $(\rho-R)$ of the roots are respectively $\frac{-\delta\beta}{C}$ and $\beta-\frac{\delta+\lambda}{C}$ †. In consequence, the following equality holds

$$\frac{\beta-R}{\beta} = \frac{\lambda}{C(\beta+\rho)} \ ^{\ddagger}.$$

Thus, it follows

$$\psi_\delta(u) = \frac{\lambda\beta}{C(\beta+\rho)^2}e^{-Ru} \Rightarrow^{\S} \psi_0(u) = \frac{\lambda}{\beta C}e^{-Ru}.$$

*. which follows from (2.9)

†. If x_1 and x_2 are roots of the second order equation $aX^2+bX+c=0$ $(a\neq 0)$, then we have $x_1x_2 = \frac{c}{a}$ and $x_1+x_2 = -\frac{b}{a}$.

‡. since we have $(\beta-R)(\beta+\rho) = \beta^2+\beta(-\beta+\frac{\delta+\lambda}{C})-\frac{\beta\delta}{C} = \frac{\beta\lambda}{C}$

§. $\delta=0 \Rightarrow \rho=0$.

2.1.8 Explicit expressions (Continued)

In the sub-section 2.1.3, we have seen that $f(x|u)$ ¶ and $f(x,y|u)$ ‖ depends on ψ_δ. So explicit expressions of the ruin probability ψ_δ implies explicit expressions of $f(x|u)$ and $f(x,y|u)$. An explicit expressions of ψ_δ can be derived from $\widehat{\psi}_\delta$ by locating its singularities. Since we have the following

$$\widehat{\psi}_\delta(\xi) = \frac{\widehat{g}(\xi) - \widehat{g}(\rho)}{(1-\widehat{g}(\xi))(\rho - \xi)},$$

derived from the Laplace transform of the renewal equation (2.6) and the definition of h. Hence, $\widehat{\psi}_\delta$ is rational if and only if $\widehat{g}$ is a rational function. From the previous equation, it follows the singularities of $\widehat{\psi}_\delta$ are exactly the roots of

$$\widehat{g}(\xi) = 1. \tag{2.15}$$

This equation can have multiple solutions if we consider the complex solutions. Note that $-R$, the adjustment coefficient is a root of (2.15), but not ρ, since $\widehat{g}(\rho) = \psi_\delta(0)$.

Using the Heaviside's expansion formula *, the authors get that

$$\psi_\delta(u) = \sum_{k=1}^{m} \lim_{\xi \to -r_k} (\xi + r_k)\widehat{\psi}_\delta(\xi) e^{\xi u},$$

where $(-r_k)_{1\leq k\leq m}$ stands for the distinct roots of (2.15), when r_k are simple roots (i.e. multiplicity equals to one). This leads to

$$\psi_\delta(u) = \sum_{k=1}^{m} \frac{\widehat{g}(-r_k) - \widehat{g}(\rho)}{-\left(\widehat{g}\right)'(-r_k)(\rho + r_k)} e^{-r_k u}, \tag{2.16}$$

using the previous expression of $\widehat{\psi}_\delta$.

2.1.9 Mixture of exponentials or the hyper-exponential

Let us consider the case where the claim sizes X are distributed according to a mixture of exponentials. That is to say

$$f_X(x) = \sum_{j=1}^{n} A_j \beta_j e^{-\beta_j x},$$

where $0 < \beta_1 < \cdots < \beta_n$, $A_i \geq 0$ and $\sum_{i=1}^{n} A_i = 1$. We have

$$\widehat{f}_X(\xi) = \sum_{j=1}^{n} \frac{A_j \beta_j}{\beta_j + \xi},$$

¶. the density of U_{τ^-} knowing $U_0 = u$.
‖. the joint density of U_{τ^-} and $|U_\tau|$ knowing $U_0 = u$.
*. cf. the appendix B.6

thus the Lundberg equation (2.11) becomes

$$\delta + \lambda - C\xi = \lambda \sum_{j=1}^{n} \frac{A_j \beta_j}{\beta_j + \xi}.$$

Let $(-r_k)_{1 \leq k \leq m}$ be the roots of the supraequation, which are supposed to be distinct and simple. We recall that

$$\widehat{g}(\xi) = \frac{\lambda(\widehat{f}_X(\xi) - \widehat{f}_X(\rho))}{C(\rho - \xi)} = \frac{\lambda}{C} \sum_{j=1}^{n} \frac{A_j \beta_j}{(\beta_j + \xi)(\beta_j + \rho)},$$

from which we have,

$$\widehat{h}(\xi) = \frac{\widehat{g}(\xi) - \widehat{g}(\rho)}{\rho - \xi} = \frac{\lambda}{C} \sum_{j=1}^{n} \frac{A_j \beta_j}{(\beta_j + \rho)^2 (\beta_j + \xi)},$$

and

$$\left(\widehat{g}\right)'(\xi) = -\frac{\lambda}{C} \sum_{j=1}^{n} \frac{A_j \beta_j}{(\beta_j + \xi)^2 (\beta_j + \rho)}.$$

Therefore we get from the relation (2.16)

$$\psi_\delta(u) = \sum_{k=1}^{m} \frac{\sum_{j=1}^{n} \frac{A_j \beta_j}{(\beta_j + \rho)^2 (\beta_j - r_k)}}{\sum_{j=1}^{n} \frac{A_j \beta_j}{(\beta_j - r_k)^2 (\beta_j + \rho)}} e^{-r_k u}. \tag{2.17}$$

This formula slightly simplifies when $\delta = 0$ (which implies that $\rho = 0$),

$$\psi_0(u) = \sum_{k=1}^{m} \frac{\sum_{j=1}^{n} \frac{A_j}{\beta_j (\beta_j - r_k)}}{\sum_{j=1}^{n} \frac{A_j}{(\beta_j - r_k)^2}} e^{-r_k u},$$

that has been known for many years. In Gerber & Dufresnes (1991*b*), we can find a proof that $\psi(u) = \sum_{k=1}^{m} C_k e^{-r_k u}$, a special case of (2.17).

2.2 Proportional reinsurance and analysis of ruin measures

In the previous section, we try to summarize the main results of the heavily dense article of Gerber & Shiu (1998). If some parts may have seemed unclear, full explanations will be given in this section, that will extend the previous one. As the overall focus of this memoir is reinsurance and ruin theory, we want to add reinsurance into the Gerber-Shiu analysis of the expected discounted penalty function in the Cramér-Lundberg model.

We focus on proportional reinsurance per risk, where the insurer keeps a of his risk, the rest is transfered to the reinsurer. So after reinsurance, the aggregate loss of the insurer is $S_t(a) = \sum_{i=1}^{N_t} X_i(a) = aS_t$. So the surplus process is

$$U_t^a = u + C(a)t - aS_t,$$

where $C(a)$ is the premium rate net of reinsurance, and $a \in]a_0, 1]$ * the retention limit.

*. cf. chapter 1

For the moment, we do not need to specify the premium principle calculation. As Gerber & Shiu (1998), we consider that $(S_t)_t$ is a compound Poisson process of parameter λ (in particular we have the independence between claim size and claim frequency).

Before finding the renewal equation, we have some properties on $X(a)$

$$F_{X(a)}(x) \overset{\triangle}{=} P(aX \leq x) = F_X(\frac{x}{a}),\ f_{X(a)}(x) = \frac{1}{a} f_X\left(\frac{x}{a}\right) \text{ and } \widehat{f}_{X(a)}(\xi) = \widehat{f}_X(a\xi).$$

2.2.1 A renewal equation for φ_a

Again we denote by $f_a(x, y, t|u)$, the joint density of $U^a_{\tau-}$, $|U^a_\tau|$ and τ knowing that $U^a_0 = u$, i.e. one might write $f^{U^a_0=u}_{U^a_{\tau-},|U^a_\tau|,\tau}(x, y, z)$. Let us notice that f_a is a defective density, since

$$\psi(u) = \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} f_a(x, y, t|u) dxdydt < 1,$$

because we suppose that $(U^a_t)_t$ has a strictly positive drift.

We temporaly drop the index δ on φ to simplify the notation. As in the previous section, we define $f_a(x, y|u)$ as the joint density $U^a_{\tau-}$ and $|U^a_\tau|$ knowing that $U^a_0 = u$. Then the expected discounted penalty function is defined by

$$\varphi_a(u) \overset{\triangle}{=} E\left[e^{-\delta\tau} w(U^a_{\tau-}, |U^a_\tau|)\mathbf{1}_{(\tau<+\infty)}/U^a_0 = u\right]. \tag{2.18}$$

To get the functional equation, we condition on interval $[0, h]$ by the fact that there is a claim or not. Remarks, that the probability there are two claims on $[0, h]$ is $o(h)$ since $(N_t)_t$ is a renewal process with continuous inter-occurence times. From (2.18), we have

$$\begin{aligned}
\varphi_a(u) &= E\left[e^{-\delta\tau} w(U^a_{\tau-}, |U^a_\tau|)\mathbf{1}_{(\tau<+\infty)}/U^a_0 = u\right] \\
&= E\left[e^{-\delta\tau} w(U^a_{\tau-}, |U^a_\tau|)\mathbf{1}_{(\tau<+\infty)}/U^a_0 = u, N_h = 0\right] P(N_h = 0) \\
&\quad + E\left[e^{-\delta\tau} w(U^a_{\tau-}, |U^a_\tau|)\mathbf{1}_{(\tau<+\infty)}/U^a_0 = u, N_h = 1\right] P(N_h = 1) \\
&= e^{-(\lambda+\delta)h} \varphi_{\delta,a}(u + C(a)h) \\
&\quad + \int_0^h \int_0^{+\infty} E\left[e^{-\delta(\zeta+t)} w(U^a_{\tau-}, |U^a_\tau|)\mathbf{1}_{(\zeta+t<+\infty)}/U^a_t = u + C(a)t - x\right] \lambda e^{-\lambda t} f_{X(a)}(x) dxdt \\
&= e^{-(\lambda+\delta)h} \varphi_a(u + C(a)h) + \int_0^h \int_0^{u+C(a)t} \varphi_a(u + C(a)t - x) \lambda e^{-(\lambda+\delta)t} f_X\left(\frac{x}{a}\right) \frac{dx}{a} dt \\
&\quad + \int_0^h \int_{u+C(a)t}^{+\infty} e^{-\delta t} w(u + C(a)t, x - C(a)t - u) \lambda e^{-\lambda t} f_X\left(\frac{x}{a}\right) \frac{dx}{a} dt.
\end{aligned}$$

Now we differentiate the previous relation with respect to h. We get *

$$\begin{aligned}\frac{\partial \varphi_a(u)}{\partial h} &= -(\lambda+\delta)e^{-(\lambda+\delta)h}\varphi_a(u+C(a)h) + e^{-(\lambda+\delta)h}C(a)\varphi_a'(u+C(a)h) \\ &\quad + \int_0^{u+C(a)h} \varphi_a(u+C(a)h-x)\lambda e^{-(\lambda+\delta)h} f_X\left(\frac{x}{a}\right)\frac{dx}{a} \\ &\quad + \int_{u+C(a)h}^{+\infty} e^{-\delta h} w(u+C(a)h, x-C(a)h-u)\lambda e^{-\lambda h} f_X\left(\frac{x}{a}\right)\frac{dx}{a}.\end{aligned}$$

By taking $h=0$, we obtain the equivalent functional equation of (2.3) when we take into account proportional reinsurance

$$0 = -(\lambda+\delta)\varphi_a(u) + C(a)\varphi_a'(u) + \int_0^u \varphi_a(u-x)\lambda f_X\left(\frac{x}{a}\right)\frac{dx}{a} + \int_u^{+\infty} w(u, x-u)\lambda f_X\left(\frac{x}{a}\right)\frac{dx}{a},$$

which is equivalent to

$$\varphi_a'(u) = \frac{\lambda+\delta}{C(a)}\varphi_a(u) - \frac{\lambda}{C(a)}\int_0^u \varphi_a(u-x) f_X\left(\frac{x}{a}\right)\frac{dx}{a} - \frac{\lambda}{C(a)}\omega_a(u), \tag{2.19}$$

where $\omega_a(u) = \int_u^{+\infty} w(u, x-u) f_X\left(\frac{x}{a}\right)\frac{dx}{a}$.

Then we multiply both sides of (2.19) by $e^{-\rho u}$ for some positive ρ. Using $\varphi_{a,\rho}(u) \stackrel{\triangle}{=} \varphi_a(u)e^{-\rho u}$ and $\varphi_{a,\rho}'(u) = \varphi_a'(u)e^{-\rho u} - \rho\varphi_{a,\rho}(u)$, we get

$$C(a)\varphi_{a,\rho}'(u) = (\lambda+\delta-C(a)\rho)\varphi_{a,\rho}(u) - \lambda\int_0^u \varphi_{a,\rho}(u-x)e^{-\rho x} f_X\left(\frac{x}{a}\right)\frac{dx}{a} - \lambda e^{-\rho u}\omega_a(u).$$

We now impose ρ to be the root of the following equation

$$\lambda+\delta-C(a)\xi = \lambda\widehat{f}_X(a\xi), \tag{2.20}$$

where the roots are $\xi_1 = \rho \geq 0$ and $\xi_2 = -R < 0$ as in Gerber & Shiu (1998). Hence, from (2.20) the integro-differential equation becomes

$$C(a)\varphi_{a,\rho}'(u) = \lambda\widehat{f}_X(a\rho)\varphi_{a,\rho}(u) - \lambda\int_0^u \varphi_{a,\rho}(u-x)e^{-\rho x} f_X\left(\frac{x}{a}\right)\frac{dx}{a} - \lambda e^{-\rho u}\omega_a(u). \tag{2.21}$$

As we want to simplify the terms with "$\varphi_{a,\rho}$" in the right-hand side, the goal is to make appear the Laplace transform of f_X. So, we integrate the right-hand side term with respect to u between 0 and $z>0$,

$$\lambda\widehat{f}_X(a\rho)\int_0^z \varphi_{a,\rho}(u)du - \lambda\int_0^z\int_0^u \varphi_{a,\rho}(u-x)e^{-\rho x} f_X\left(\frac{x}{a}\right)\frac{dx}{a}du - \lambda\int_0^z e^{-\rho u}\omega_a(u)du.$$

By changing the variable and the order of integration, we have

$$\begin{aligned}\int_0^z\int_0^u \varphi_{a,\rho}(u-x)e^{-\rho x} f_X\left(\frac{x}{a}\right)\frac{dx}{a}du &= \int_0^z\int_0^u \varphi_{a,\rho}(y)e^{-\rho(u-y)} f_X\left(\frac{u-y}{a}\right)\frac{dy}{a}du \\ &= \int_0^z \varphi_{a,\rho}(y)\int_y^z e^{-\rho(u-y)} f_X\left(\frac{u-y}{a}\right)\frac{du}{a}dy \\ &= \int_0^z \varphi_{a,\rho}(y)\int_0^{z-y} e^{-\rho x} f_X\left(\frac{x}{a}\right)\frac{dx}{a}dy.\end{aligned}$$

*. cf. the appendix B.7 on differentiation of functions defined by integrals.

Since $\int_0^u e^{-\rho x} f_X\left(\frac{x}{a}\right)\frac{dx}{a} = \widehat{f}_X(a\rho) - \int_u^{+\infty} e^{-\rho x} f_X\left(\frac{x}{a}\right)\frac{dx}{a}$, we get that

$$\int_0^z \int_0^u \varphi_{a,\rho}(u-x)e^{-\rho x} f_X\left(\frac{x}{a}\right)\frac{dx}{a}du \quad = \quad \int_0^z \varphi_{a,\rho}(y)\left(\widehat{f}_X(a\rho) - \int_{z-y}^{+\infty} e^{-\rho x} f_X\left(\frac{x}{a}\right)\frac{dx}{a}\right)dy.$$

Therefore, integrating the left-hand side of (2.21) and using the latter developments on the integration of the righ-hand side of (2.21), we get

$$C(a)(\varphi_{a,\rho}(z) - \varphi_a(0)) = \lambda \int_0^z \varphi_{a,\rho}(y) \int_{z-y}^{+\infty} e^{-\rho x} f_X\left(\frac{x}{a}\right)\frac{dx}{a}dy - \lambda \int_0^z e^{-\rho u}\omega_a(u)du, \tag{2.22}$$

when $z \to +\infty$, we get

$$0 = \varphi_a(0) - \frac{\lambda}{C(a)}\int_0^{+\infty} e^{-\rho u}\omega_a(u)du.$$

Since $\widehat{\omega}_a(\rho) - \int_0^z e^{-\rho u}\omega_a(u)du = \int_z^{+\infty} e^{-\rho u}\omega_a(u)du$, we get from (2.22)

$$C(a)\varphi_{a,\rho}(z) = \lambda \int_0^z \varphi_{a,\rho}(y)\int_{z-y}^{+\infty} e^{-\rho x} f_X\left(\frac{x}{a}\right)\frac{dx}{a}dy + \lambda\int_z^{+\infty} e^{-\rho u}\omega_a(u)du,$$

multiplying by $e^{\rho z}$, it follows

$$\varphi_a(z) = \frac{\lambda}{C(a)}\int_0^z \varphi_a(y)\int_{z-y}^{+\infty} e^{-\rho(y+x-z)} f_X\left(\frac{x}{a}\right)\frac{dx}{a}dy + \frac{\lambda}{C(a)}\int_z^{+\infty} e^{-\rho(u-z)}\omega_a(u)du.$$

Finally, the renewal equation which extends (2.6) is

$$\varphi_{\delta,a} = \varphi_{\delta,a} * g_a + h_a, \tag{2.23}$$

where $*$ stands for the convolution product,

$$g_a(z) = \frac{\lambda}{C(a)}\int_z^{+\infty} e^{-\rho(x-z)} f_X\left(\frac{x}{a}\right)\frac{dx}{a},$$

and

$$h_a(z) = \frac{\lambda}{C(a)}\int_z^{+\infty} e^{-\rho(u-z)}\omega_a(u)du.$$

2.2.2 Solving the renewal equation with the Laplace transform

The usual way to solve (2.23) is to take its Laplace transform. So, the equation (2.23) becomes

$$\widehat{\varphi}_{\delta,a} = \widehat{\varphi}_a\widehat{g}_a + \widehat{h}_a, \text{ which is equivalent to } \widehat{\varphi}_a = \frac{\widehat{h}_a}{1-\widehat{g}_a}.$$

Let us work on the Laplace transform of g_a and h_a. We have

$$\begin{aligned}
\widehat{g}_a(\xi) &= \frac{\lambda}{C(a)} \int_0^{+\infty} e^{-\xi z} \int_z^{+\infty} e^{-\rho(x-z)} f_X\left(\frac{x}{a}\right) \frac{dx}{a} dz \\
&= \frac{\lambda}{C(a)} \int_0^{+\infty} e^{-\rho x} \int_0^{x} e^{-\xi z} e^{\rho z} dz f_X\left(\frac{x}{a}\right) \frac{dx}{a} \\
&= \frac{\lambda}{C(a)} \int_0^{+\infty} e^{-\rho x} \left(\frac{e^{(\rho-\xi)x}-1}{\rho-\xi}\right) f_X\left(\frac{x}{a}\right) \frac{dx}{a} \\
&= \frac{\lambda}{C(a)(\rho-\xi)} \left(\widehat{f}_{X(a)}(\xi) - \widehat{f}_{X(a)}(\rho)\right),
\end{aligned}$$

and

$$\begin{aligned}
\widehat{h}_a(\xi) &= \frac{\lambda}{C(a)} \int_0^{+\infty} e^{-\xi z} \int_z^{+\infty} e^{-\rho(u-z)} \omega_a(u) du dz \\
&= \frac{\lambda}{C(a)} \int_0^{+\infty} \omega_a(u) e^{-\xi u} \int_0^{u} e^{-(\xi-\rho)z} dz du \\
&= \frac{\lambda}{C(a)} \int_0^{+\infty} \omega_a(u) e^{-\xi u} \left(\frac{e^{(\rho-\xi)u}-1}{\rho-\xi}\right) du \\
&= \frac{\lambda}{C(a)(\rho-\xi)} \left(\widehat{\omega}_a(\xi) - \widehat{\omega}_a(\rho)\right),
\end{aligned}$$

where $\omega_a(u) = \int_u^{+\infty} w(u, x-u) f_X\left(\frac{x}{a}\right) \frac{dx}{a}$. Subsequently, the Laplace transform of the Gerber-Shiu function is

$$\widehat{\varphi}_{\delta,a} = \frac{\lambda(\widehat{\omega}_a(\xi) - \widehat{\omega}_a(\rho))}{C(a)(\rho-\xi) + \lambda \widehat{f}_{X(a)}(\rho) - \lambda \widehat{f}_{X(a)}(\xi)}.$$

Finally, using the fact ρ is a root of the Lundberg equation (2.20), we have

$$\widehat{\varphi}_{\delta,a} = \frac{\lambda(\widehat{\omega}_a(\xi) - \widehat{\omega}_a(\rho))}{-\xi C(a) + \lambda + \delta - \lambda \widehat{f}_{X(a)}(\xi)}. \tag{2.24}$$

If we consider $w(x,y) = 1$, we have

$$\omega_a(u) = \int_u^{+\infty} f_X\left(\frac{x}{a}\right) \frac{dx}{a} = \overline{F}_X\left(\frac{u}{a}\right),$$

hence, $\widehat{\omega}_a(\xi) = a\widehat{\overline{F}}_X(a\xi) = \frac{1}{\xi} - \frac{1}{\xi}\widehat{f}_X(a\xi)$, using $\widehat{f'}(\xi) = \xi \widehat{f}(\xi) - f(0)$ *. Since $\lambda + \delta - C(a)\rho = \lambda \widehat{f}_X(a\rho)$, we get

$$\widehat{h}_a(\xi) = \frac{\lambda}{C(a)(\rho-\xi)} \left(\frac{1}{\xi} - \frac{1}{\xi}\widehat{f}_X(a\xi) + \frac{\delta}{\lambda\rho} - \frac{C(a)}{\lambda}\right).$$

Therefore, with $w(x,y) = 1$, the Laplace transform of φ (2.24) becomes

$$\widehat{\varphi}_{\delta,a} = \frac{\frac{\lambda}{\xi}(1 - \widehat{f}_X(a\xi)) + \frac{\delta}{\rho} - C(a)}{\lambda(1 - \widehat{f}_X(a\xi)) + \delta - C(a)\xi},$$

*. when the Laplace transform exists s, cf. appendix B.6

which extends (2.8). And when $\delta = 0$ (that implies $\rho = 0$), we have

$$\widehat{\psi}_a = \frac{\frac{\lambda}{\xi}(1 - \widehat{f}_X(a\xi)) + \lambda a E[X]}{\lambda(1 - \widehat{f}_X(a\xi)) - C(a)\xi}.$$

If we use Dickson (1998), we can derive the Laplace transform of the solution by taking the Laplace transform of (2.19)

$$\widehat{\varphi}_{\delta,a}(\xi) = \frac{C(a)\varphi_{\delta,a}(0) - \lambda\widehat{\omega}_a(\xi)}{C(a)\xi - (\lambda + \delta) + \lambda\widehat{f}_{X(a)}(\xi)},$$

since $\widehat{f'}(\xi) = \xi\widehat{f}(\xi) - f(0)$ †. But to obtain the equation (2.24), we need to know $0 = \varphi_a(0) - \frac{\lambda}{C(a)}\int_0^{+\infty} e^{-\rho u}\omega_a(u)du$, which was obtained as a limit of the functional equation of $\varphi_{a,\rho}$.

2.2.3 Martingales

As presented in Gerber & Shiu (1998), the roots of the Lundberg equation has a very pleasant property on the process $(V_{\xi,t})_t$ defined by $\left(e^{\delta t + \xi U_t^a}\right)_t$. For the same reason used in 2.1.6, the process V is a martingale if and only if

$$E\left[e^{-\delta t + \xi U_t^a} / U_0^a = u\right] = e^{\xi u}.$$

Once again, we use the fact that S_t is a coumpound Poisson process of moment generating function $M_{S_t}(\xi) = G_{N_t}(M_X(\xi))$, where G_{N_t} stands for the probability generating function. Thus, the "martingale condition" yields to

$$-\delta t + \xi(u + C(a)t) + \lambda t(\widehat{f}_X(a\xi) - 1) = \xi u,$$

which is the Lundberg equation (2.20). Hence, $(V_{\rho,t})_t$ and $(V_{-R,t})_t$ are martingales. We are now able to define the ruin probability when $\delta > 0$,

$$\psi_{\delta,a} = E\left[e^{-\delta\tau + \rho U_\tau^a}\mathbf{1}_{(\tau<+\infty)} / U_0^a = u\right].$$

Since for all $0 \leq t \leq \tau$, $\delta t + RU_t^a \geq 0$ and hence $0 \leq e^{-\delta\tau - RU_\tau^a} \leq 1$, we can use the optional sampling theorem * with the stopping time τ,

$$E\left[e^{-\delta\tau - RU_\tau^a}\mathbf{1}_{(\tau<+\infty)} / U_0^a = u\right] = e^{-Ru}, \tag{2.25}$$

which generalizes (2.14). In the same way, we can have the corresponding relation with the stopping time T_x and ρ.

Finally, as in Gerber & Shiu (1998), we have an explicit expression of the Laplace transform of the finite time ruin probabilities. Indeed, the finite time ruin probability $\psi_a(u,t)$ is defined as $P(\tau < t / U_0^a = u)$, and its Laplace transform

$$\widehat{\psi}_a(\xi, \delta) = \int_0^{+\infty}\int_0^{+\infty} e^{-\xi u - \delta t}\psi_a(u,t)dudt.$$

†. cf. appendix B.6
*. cf. the appendix B.5

We have the following relation between the discounted penalty function and $\widehat{\psi}_a(\xi,\delta) : \frac{\varphi_{\delta,a}(\xi)}{\delta}$ when $w(x,y)=1$. Thus, we have

$$\widehat{\psi}_a(\xi,\delta) = \frac{\frac{\lambda}{\xi}(1-\widehat{f}_X(a\xi)) + \frac{\delta}{\rho} - C(a)}{\delta(\lambda(1-\widehat{f}_X(a\xi)) + \delta - C(a)\xi)}.$$

Using the survival probability $\phi_a(u,t) = 1-\psi_a(u,t)$ (i.e. $\widehat{\phi}_a(\xi,\delta) = \frac{1}{\delta\xi} - \widehat{\psi}_a(\xi,\delta)$ †), one can obtain the Laplace tranform of $\widehat{\phi}_a$.

2.2.4 Some explicit results on the surplus prior ruin, the deficit at ruin and the ruin probability

In this sub-section, we give our interest on the joint density of the surplus prior ruin $U^a_{\tau^-}$ and the deficit at ruin $|U^a_\tau|$. At the beginning of this section, we denote by f_a the joint density of the surplus prior ruin $U^a_{\tau^-}$, the deficit at ruin $|U^a_\tau|$ and the ruin time τ. Here are some properties of f_a :

- for $x > u + C(a)t$, $f_a(x,y,t|u) = 0$, since the event $\{U^a_{\tau^-} > u + C(a)\tau\}$ is equivalent to $\{0 > aS_\tau\}$;
- when $x = u + C(a)t$, we have that $\tau = W_1$ (ruin at the first claim), which implies that the differential $f_a(u+C(a)t, y, t|u)dydt = \lambda e^{-\lambda t} f_{X(a)}(u + C(a)t + y)dydt$;
- $f_a(x,y,t|u) = \left(\int_0^{+\infty} f_a(x,z,t|u)dz\right) \frac{f_{X(a)}(x+y)}{\overline{F}_{X(a)}(x)}$, by conditioning on $U^a_{\tau^-} = x$ and $\tau = t$.

Let $f_a(x,y|u)$ be the joint density of $U^a_{\tau^-}$ and $|U^a_\tau|$ defined as $f_a(x,y|u) = \int_0^{+\infty} e^{-\delta t} f_a(x,y,t|u)dt$. Thus, we have

$$\begin{aligned}\int_0^{+\infty} e^{-\delta t} f_a(x,y,t|u)dt &= \int_0^{+\infty} e^{-\delta t}\left(\int_0^{+\infty} f_a(x,z,t|u)dz\right) dt \frac{f_{X(a)}(x+y)}{\overline{F}_{X(a)}(x)} \\ \Leftrightarrow f_a(x,y|u) &= f_a(x|u)\frac{f_{X(a)}(x+y)}{\overline{F}_{X(a)}(x)}. \end{aligned} \tag{2.26}$$

The last relation (2.26) (which extends (2.9)) will have strong consequences in this section.

Using the renewal equation (2.23) in $u=0$, we have $\varphi_{\delta,a} = h_a(0)$, which is equivalent to

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty} w(x,y)f_a(x,y|0)dxdy &= \frac{\lambda}{C(a)}\int_0^{+\infty} e^{-\rho x}\int_x^{+\infty} w(x,u-x)f_{X(a)}(u)dudx \\ \Leftrightarrow \int_0^{+\infty}\int_0^{+\infty} w(x,y)f_a(x,y|0)dxdy &= \int_0^{+\infty}\int_0^{+\infty} w(x,y)\frac{\lambda e^{-\rho x}}{C(a)} f_{X(a)}(x+y)dydx.\end{aligned}$$

Hence,

$$f_a(x,y|0) = \frac{\lambda e^{-\rho x}}{aC(a)} f_X\left(\frac{x+y}{a}\right) \Rightarrow f_a(x|0) = \frac{\lambda e^{-\rho x}}{C(a)}\overline{F}_X\left(\frac{x}{a}\right). \tag{2.27}$$

This extends the corresponding relation (2.10) without reinsurance and where $\overline{F}_X$ stands for the survival function. Integrating the last expression with respect to x, it yields to $\varphi_{\delta,a}(0) = \frac{\lambda}{C(a)} a\widehat{\overline{F}}_X(x)$

†. cf. appendix B.6

for $w(x,y)=1$. Using $a\widehat{\overline{F}}_X(a\xi)=\frac{1}{\xi}-\frac{1}{\xi}\widehat{f}_X(a\xi)$ * and the fact ρ is the solution of the Lundberg equation (2.20), we get

$$\varphi_{\delta,a}(0)=\frac{C(a)\rho-\delta}{C(a)\rho}\text{ for }\delta\geq 0,$$

with the special case where $\delta=0$

$$\varphi_{\delta,a}(0)=\frac{a\lambda E[X]}{C(a)}.$$

The last two equations extends what was found without reinsurance.

As seen in the first section, there is a relation between $f_a(x|u)$ * and $f_a(x|0)$ (so $f_a(x,y|u)$ † and $f_a(x,y|0)$). As defined in the sub-section 2.2.3, the ruin probability for all $\delta\geq 0$ is defined by $\psi_\delta(u)=E\left[e^{-\delta\tau+\rho U^a_\tau}\mathbf{1}_{(\tau<+\infty)}/U^a_0=u\right]$. With this definition, we have the following relations

$$f_a(x|u)=\begin{cases} f_a(x|0)\frac{e^{\rho u}-\psi_\delta(u)}{1-\psi_\delta(0)} & \text{if } x>u\geq 0\\ f_a(x|0)\frac{e^{\rho x}\psi_\delta(u-x)-\psi_\delta(u)}{1-\psi_\delta(0)} & \text{if } 0<x\leq u\end{cases},$$

where $f_a(x|0)$ is given by (2.27). An expression of $f_a(x,y|u)$ in function of $f_a(x,y|0)$ can also be derived

$$f_a(x,y|u)=\begin{cases} f_a(x,y|0)\frac{e^{\rho u}-\psi_\delta(u)}{1-\psi_\delta(0)} & \text{if } x>u\geq 0\\ f_a(x,y|0)\frac{e^{\rho x}\psi_\delta(u-x)-\psi_\delta(u)}{1-\psi_\delta(0)} & \text{if } 0<x\leq u\end{cases},$$

where $f_a(x,y|0)$ is given by (2.27). The demonstration, which uses stopping times defined as a upcrossing of barrier levels, has been put in appendix B.8. In the previous results on the ruin probability, we don't explicitely put the link with the retention rate a (because of the numerous indexes), but they depends on a.

Finally, we have an explicit expressions of the ruin probability $\psi_{\delta,a}$ in the following special case. As used by Gerber and Shiu, an explicit expression of $\psi_{\delta,a}$ can derived from locating the singularitiesof its Laplace transform. When the penalty function $w(x,y)=e^{-\rho y}$ (i.e. φ is the ruin probability), we have

$$\omega_a(u)=\int_u^{+\infty}e^{-\rho(x-u)}f_X(\frac{x}{a})\frac{dx}{a}=g_a(u)\frac{C(a)}{\lambda}.$$

Using the definition of $\widehat{h}_a(\xi)$, we get

$$\widehat{h}_a(\xi)=\int_0^{+\infty}e^{-\xi z}\int_z^{+\infty}e^{-\rho(u-z)}g_a(z)dudz=\frac{\widehat{g}_a(\xi)-\widehat{g}_a(\rho)}{\rho-\xi},$$

consequently, it yields to

$$\widehat{\psi}_{\delta,a}(\xi)=\frac{\widehat{g}_a(\xi)-\widehat{g}_a(\rho)}{(1-\widehat{g}_a(\xi))(\rho-\xi)},$$

which implies that the singularities of $\widehat{\psi}_{\delta,a}$ are the roots of $\widehat{g}_a(\xi)=1$.

∗. cf. appendix B.6, that $\widehat{f'}(s)=s\widehat{f}(s)-f(0)$

∗. the density of $U^a_{\tau-}$ knowing $U^a_0=u$.

†. the density of $U^a_{\tau-}$ and $|U^a_\tau|$ knowing $U^a_0=u$.

Using the Heaviside's expansion formula ‡, we get

$$\psi_\delta(u) = \sum_{k=1}^{m} \lim_{\xi \to -r_k} (\xi + r_k)\widehat{\psi}_{\delta,a}(\xi)e^{\xi u},$$

where $(-r_{a,k})_{1\leq k\leq m}$ stands for the distinct roots of (2.15), when $r_{a,k}$ are supposed to be simple roots (i.e. their multiplicity equals to one). This leads to

$$\psi_\delta(u) = \sum_{k=1}^{m} \frac{1}{-\left(\widehat{g}_a\right)'(-r_{a,k})}\widehat{h}_a(-r_{a,k})e^{-r_{a,k}u}. \tag{2.28}$$

2.2.5 Asymptotic results

As done in Gerber & Shiu (1998), we derived some asymptotic results using the key renewal theorem ∗:

$$\varphi_{\delta,a}(u) \underset{+\infty}{\sim} \frac{\widehat{h_a}(-R)}{-\left(\widehat{g}_a\right)'(-R)}e^{-Ru},$$

where $\widehat{h_a}(-R) = \frac{\lambda}{C(a)(\rho+R)}\left(\widehat{\omega}_a(-R) - \widehat{\omega}_a(\rho)\right)$ and $\left(\widehat{g}_a\right)'(-R) = \frac{\lambda}{C(a)(\rho+R)}\left(\left(\widehat{f}_X\right)'(-aR) + \frac{C(a)}{\lambda}\right)$. It yields to

$$\varphi_{\delta,a}(u) \underset{+\infty}{\sim} \frac{\lambda(\widehat{\omega}_a(-R) - \widehat{\omega}_a(\rho))}{-\lambda\left(\widehat{f}_X\right)'(-aR) - C(a)}e^{-Ru}. \tag{2.29}$$

In consequence, the equivalent (2.29) generalizes those of sub-section 2.1.5, as those that follows. Thanks to the definition of ω_a and the definition of the expected discounted penalty function (2.18), we can identify the equivalent of $f_a(x, y|u)$. Indeed, we have

$$\widehat{\omega}_a(-R) - \widehat{\omega}_a(\rho) = \int_0^{+\infty}\int_0^{+\infty} w(x, y)(e^{Rx} - e^{-\rho x})f_{X(a)}(x + y)dxdy.$$

Hence, we obtain

$$f_a(x, y|u) \underset{+\infty}{\sim} \frac{\lambda(e^{Rx} - e^{-\rho x})f_X\left(\frac{x+y}{a}\right)}{-\lambda\left(\widehat{f}_X\right)'(-aR) - C(a)}e^{-Ru},$$

by integrating with respect to y, we get

$$f_a(x|u) \underset{+\infty}{\sim} \frac{\lambda(e^{Rx} - e^{-\rho x})\overline{F}_X\left(\frac{x}{a}\right)}{-\lambda\left(\widehat{f}_X\right)'(-aR) - C(a)}e^{-Ru}.$$

And finally, we take the special case where $w(x, y) = 1$, the equivalent (2.29) becomes

$$\varphi_{\delta,a}(u) \underset{+\infty}{\sim} \frac{\delta(1/R + 1/\rho)}{-\lambda\left(\widehat{f}_X\right)'(-aR) - C(a)}e^{-Ru},$$

using the value of $\widehat{h}_a(\xi)$ in $-R$ when $w(x, y) = 1$ developped in sub-section 2.2.2.

‡. cf. appendix B.6

∗. cf. appendix B.2

2.2.6 Exponentially distributed claim sizes

We consider in this sub-section that the claim sizes X follows an exponential distribution $\mathcal{E}(\beta)$. First, the Lundberg equation (2.20) becomes a second order equation

$$aC(a)\xi^2 + (C(a)\beta - a(\delta+\lambda))\xi - \beta\delta = 0. \tag{2.30}$$

The discriminant is $\Delta_a = (a(\delta+\lambda) - C(a)\beta)^2 + 4a\beta\delta C(a)$, which is positive $\forall a \in]a_0, 1]$. Thus, the roots ρ and $-R$ are

$$\rho = \frac{a(\delta+\lambda) - C(a)\beta + \sqrt{\Delta_a}}{2C(a)a} \quad \text{and} \quad R = \frac{C(a)\beta - a(\delta+\lambda) + \sqrt{\Delta_a}}{2C(a)a}. \tag{2.31}$$

Since $\widehat{f}_{X(a)}(\xi) = \frac{\beta}{\beta+a\xi}$, the relation (2.24) becomes

$$\widehat{\varphi}_{\delta,a} = \frac{\lambda(\widehat{\omega}_a(\xi) - \widehat{\omega}_a(\rho))}{-\xi C(a) + \lambda + \delta - \lambda\frac{\beta}{\beta+a\xi}}.$$

In the special case where the penalty function $w(x,y) = w(y)$, we have $\omega_a(z) = \frac{\beta}{a}e^{-\frac{\beta z}{a}}\widehat{w}\left(\frac{\beta}{a}\right)$. Thus, we obtain

$$\widehat{\omega}_a(\xi) = \frac{\frac{\beta}{a}}{\xi + \frac{\beta}{a}}\widehat{w}\left(\frac{\beta}{a}\right).$$

Therefore, it yields that

$$\widehat{\varphi}_{\delta,a}(\xi) = \frac{\lambda\beta\widehat{w}\left(\frac{\beta}{a}\right)}{a\xi(\lambda+\delta-C(a)) + \delta\beta} \times \frac{\rho - \xi}{\beta/a + \rho}.$$

One way to obtain explicit of ruin probability $\psi_{0,a}$ is to brutally invert the previous Laplace transform when $w(x,y) = 1$ and $\delta = 0$. This was done in the previous section with $a = 1$. We will use the method, that Gerber and Shiu used in their article. Applying the result (2.25) of the martingale part, we have

$$E\left[e^{-\delta\tau - RU_\tau^a}\mathbf{1}_{(\tau<+\infty)}/U_0^a = u\right] = e^{-Ru}.$$

From the explicit results part, we have

$$f_a(x,y|u) = f_a(x|u)\frac{f_{X(a)}(x+y)}{\overline{F}_{X(a)}(x)} = f_a(x|u)\frac{\beta}{a}e^{-\frac{\beta y}{a}} = f_a(x|u)f_{X(a)}(y).$$

Thanks to the independence between $U_{\tau-}^a$ and $|U_\tau^a|$ (cf. the supra-relation), we have the following interesting result

$$\begin{aligned}
E\left[e^{-\delta\tau - RU_\tau^a}\mathbf{1}_{(\tau<+\infty)}/U_0^a = u\right] &= \int_0^{+\infty}\int_0^{+\infty} e^{-R(-y)}e^{-\delta\tau}f_a(x,y,t|u)dtdxdy \\
e^{-Ru} &= \int_0^{+\infty}\int_0^{+\infty} e^{Ry}f_a(x,y|u)dxdy \\
e^{-Ru} &= \int_0^{+\infty} f_a(x|u)dx \int_0^{+\infty} e^{Ry}f_{X(a)}(y)dy \\
e^{-Ru} &= \widehat{f}_X(-aR) \times E\left[e^{-\delta\tau}\mathbf{1}_{(\tau<+\infty)}/U_0^a = u\right].
\end{aligned}$$

Hence,

$$E\left[e^{-\delta\tau}\mathbf{1}_{(\tau<+\infty)}/U_0^a=u\right]=\frac{\beta-aR}{\beta}e^{-Ru}.$$

If we do the same method with $E\left[e^{-\delta\tau}w(|U_\tau^a|)\mathbf{1}_{(\tau<+\infty)}/U_0^a=u\right]$, we get an explicit expression of $\varphi_{\delta,a}$ (when $w(x,y)=w(y)$ and $X\sim\mathcal{E}(\beta)$)

$$\varphi_{\delta,a}(u)=\widehat{w}\left(\frac{\beta}{a}\right)\left(\frac{\beta}{a}-R\right)e^{-Ru}, \tag{2.32}$$

with R given by (2.31).

Now, we have just to take $w(y)=e^{-\rho y}$ (in (2.32)) in order to have an explicit of the ruin probability $\psi_{\delta,a}(u)$.

$$\psi_{\delta,a}(u)=\frac{\beta-aR}{\beta+a\rho}e^{-Ru},$$

since $\widehat{w}\left(\frac{\beta}{a}\right)=\frac{1}{\frac{\beta}{a}+\rho}$. As $-R$ and ρ are roots of (2.31), we have

$$-R\rho=\frac{-\delta\beta}{C(a)a}\quad\text{and}\quad\rho-R=\frac{\beta}{a}-\frac{\delta+\lambda}{C(a)},\quad\text{hence}\quad(\frac{\beta}{a}-R)(\frac{\beta}{a}+\rho)=\frac{\beta\lambda}{aC(a)}.$$

Therefore, we have

$$\psi_{\delta,a}(u)=\frac{\beta\lambda}{aC(a)(\frac{\beta}{a}+\rho)^2}e^{-Ru}\;\Rightarrow\;\psi_{0,a}(u)=\frac{a\lambda}{C(a)\beta}e^{-Ru}, \tag{2.33}$$

with ρ and R given by (2.31) (and $R(\delta=0)=\frac{\beta}{a}-\frac{\lambda}{C(a)}$).

Furthermore, the whole sub-section 2.2.4 becomes in the case of exponentially distributed claim sizes

$$f_a(x,y|0)=\frac{\lambda e^{-\rho x}}{aC(a)}\beta e^{-\beta\frac{x+y}{a}}\;\Rightarrow\;f_a(x|0)=\frac{\lambda e^{-\rho x}}{C(a)}e^{-\beta\frac{x}{a}}.$$

The following terms are used in the expression of densities $f_a(x,y|u)$ and $f_a(x|u)$ *

$$\frac{e^{\rho u}-\psi_\delta(u)}{1-\psi_\delta(0)}=\frac{(\beta+a\rho)e^{\rho u}-(\beta-aR)e^{-Ru}}{a(\rho+R)}\quad\text{and}\quad\frac{e^{\rho x}\psi_\delta(u-x)-\psi_\delta(u)}{1-\psi_\delta(0)}=\frac{(\beta-aR)e^{-Ru}}{a(\rho+R)}\left(e^{(\rho+R)x}-1\right).$$

Thus, we get

$$f_a(x|u)=\begin{cases}\frac{\lambda}{C(a)a(\rho+R)}e^{-\beta\frac{x}{a}}e^{-\rho x}\left((\beta+a\rho)e^{\rho u}-(\beta-aR)e^{-Ru}\right) & \text{if } x>u\geq 0\\ \frac{\lambda(\beta-aR)}{C(a)a(\rho+R)}e^{-\beta\frac{x}{a}}\left(e^{Rx}-e^{-\rho x}\right)e^{-Ru} & \text{if } 0<x\leq u\end{cases},$$

and

$$f_a(x,y|u)=\begin{cases}\frac{\lambda\beta}{C(a)a^2(\rho+R)}e^{-\beta\frac{x+y}{a}}e^{-\rho x}\left((\beta+a\rho)e^{\rho u}-(\beta-aR)e^{-Ru}\right) & \text{if } x>u\geq 0\\ \frac{\lambda\beta(\beta-aR)}{C(a)a^2(\rho+R)}e^{-\beta\frac{x+y}{a}}\left(e^{Rx}-e^{-\rho x}\right)e^{-Ru} & \text{if } 0<x\leq u\end{cases}.$$

Finally, we are able to derive the equivalents discussed in the previous sub-section. The equivalent of $f_a(x,y|u)$ and $f_a(x|u)$ can be directly derived from the expressions above when $0<x\leq u$, since the only term depending on u in those expressions is the exponential bound e^{-Ru}. The equivalent of $\varphi_{\delta,a}(u)$, given in the previous sub-section, is a special case of the relation (2.32). The equivalent are not particularly useful since we have explicit expressions.

*. cf. sub-section 2.2.4

2.2.7 Hyper-exponentially distributed claim sizes

In this sub-section, we analyze the case where claim size distribution is hyper-exponential (also called a mixture of exponential in the literature). Using the different characterizations of the mixture of exponential recalled in the sub-section 2.1.9, and the properties of claim size distribution with proportional reinsurance (listed at the beginning of this section), we have

$$f_{X(a)}(x) = \sum_{j=1}^{n} \frac{A_j \beta_j}{a} e^{-\frac{\beta_j}{a} x},$$

where $0 < \beta_1 < \cdots < \beta_n$, $A_i > 0$ and $\sum_{i=1}^{n} A_i = 1$. We have

$$\widehat{f}_{X(a)}(\xi) = \sum_{j=1}^{n} \frac{A_j \beta_j}{\beta_j + a\xi}.$$

Therefore, the Lundberg equation (2.30) becomes

$$\delta + \lambda - C\xi = \lambda \sum_{j=1}^{n} \frac{A_j \beta_j}{\beta_j + a\xi}.$$

Let $(-r_{k,a})_{1 \le k \le m}$ be the roots of the supraequation, which are supposed to be distinct and simple. We recall that

$$\widehat{g}_a(\xi) = -\frac{\lambda(\widehat{f}_{X(a)}(\xi) - \widehat{f}_{X(a)}(\rho))}{C(\rho - \xi)} = \frac{a\lambda}{C} \sum_{j=1}^{n} \frac{A_j \beta_j}{(\beta_j + a\xi)(\beta_j + a\rho)},$$

Using the Laplace transform of $f_{X(a)}$, we have

$$\widehat{h}_a(\xi) = \frac{\widehat{g}(\xi) - \widehat{g}(\rho)}{\rho - \xi} = \frac{a^2\lambda}{C} \sum_{j=1}^{n} \frac{A_j \beta_j}{(\beta_j + a\rho)^2(\beta_j + a\xi)},$$

and

$$\left(\widehat{g}_a\right)'(\xi) = -\frac{a^2\lambda}{C} \sum_{j=1}^{n} \frac{A_j \beta_j}{(\beta_j + a\xi)^2(\beta_j + a\rho)}.$$

Therefore we get from the relation (2.28)

$$\psi_{\delta,a}(u) = \sum_{k=1}^{m} \frac{\sum_{j=1}^{n} \frac{A_j \beta_j}{(\beta_j + a\rho)^2(\beta_j - a r_{k,a})}}{\sum_{j=1}^{n} \frac{A_j \beta_j}{(\beta_j - a r_{k,a})^2(\beta_j + a\rho)}} e^{-r_{k,a} u}. \tag{2.34}$$

This formula slightly simplifies when $\delta = 0$ (which implies that $\rho = 0$),

$$\psi_{0,a}(u) = \sum_{k=1}^{m} \frac{\sum_{j=1}^{n} \frac{A_j}{\beta_j(\beta_j - a r_k)}}{\sum_{j=1}^{n} \frac{A_j}{(\beta_j - a r_k)^2}} e^{-r_k u}.$$

2.3 Impact of proportional reinsurance on the surplus prior to ruin and the deficit at ruin

In this section, we will give our attention on the surplus prior to ruin and the deficit at ruin. Numerical applications of these two ruin measures are carried out.

2.3.1 Exponentially distributed claim size

We suppose that the claim size X follows an exponential distribution $\mathcal{E}(\beta)$. We know from the previous section that the density of U_{τ^-} and the density of $|U_\tau|$ knowing $U_0^a = u$ are

$$f_a(x|u) = \begin{cases} \frac{\lambda}{C(a)a(\rho+R)} e^{-\beta\frac{x}{a}} e^{-\rho x}\left((\beta+a\rho)e^{\rho u} - (\beta - aR)e^{-Ru}\right) & \text{if } x > u \geq 0 \\ \frac{\lambda(\beta-aR)}{C(a)a(\rho+R)} e^{-\beta\frac{x}{a}}\left(e^{Rx} - e^{-\rho x}\right)e^{-Ru} & \text{if } 0 < x \leq u \end{cases},$$

and

$$f_a(y|u)^* = \begin{cases} \frac{\lambda\beta}{C(a)a(\beta+a\rho)(\rho+R)} e^{-\beta\frac{y}{a}}\left((\beta+a\rho)e^{\rho u} - (\beta - aR)e^{-Ru}\right) & \text{if } x > u \geq 0 \\ \frac{\lambda\beta}{C(a)(\beta+a\rho)} e^{-\beta\frac{y}{a}} e^{-Ru} & \text{if } 0 < x \leq u \end{cases}.$$

Note that these notations are confusing, because $f_a(x|u)$ denotes the density of the surplus prior to ruin U_{τ^-}, and $f_a(y|u)$ the one of the deficit at ruin $|U_\tau|$.

In the numerical applications, we suppose that $\delta = 0$ (which implies that $\rho = 0$) and the premium is calculated according to the expected value principle †. We denote by $\eta = 0.2$ and $\eta_R = 0.3$ the respective loading coefficients of the insurer and the reinsurer. And we set $\beta = 2$ and $\lambda = 1$.

As expected, the reinsurance greatly mitigates the risk in the sense, the probability, that the surplus prior to ruin is "big", is smaller with reinsurance than without. Hence, as shown in figure 2.1, the surplus prior to ruin is concentrated near its mean ‡ with $a = 0.5$, whereas $a = 1$ (no reinsurance), the tail of the distribution of the surplus prior to ruin is bigger.

Second, we observe the same effect of reinsurance on the deficit at ruin in figure 2.2. Note that, the density of the deficit at ruin is an exponential function, unlike the surplus prior to ruin, which is a combination of exponential functions. Finally, we notice logically that when the initial capital u increases, both the surplus prior to ruin and the deficit at ruin decreases.

∗. got by integrating $f_a(x, y|u)$ w.r.t. x.

†. cf. the equation (1.1) of the previous chapter

‡. which is $\frac{\lambda}{C(a)R}(1 - e^{-Ru})$, obtained by integration of $f_a(x|u)$.

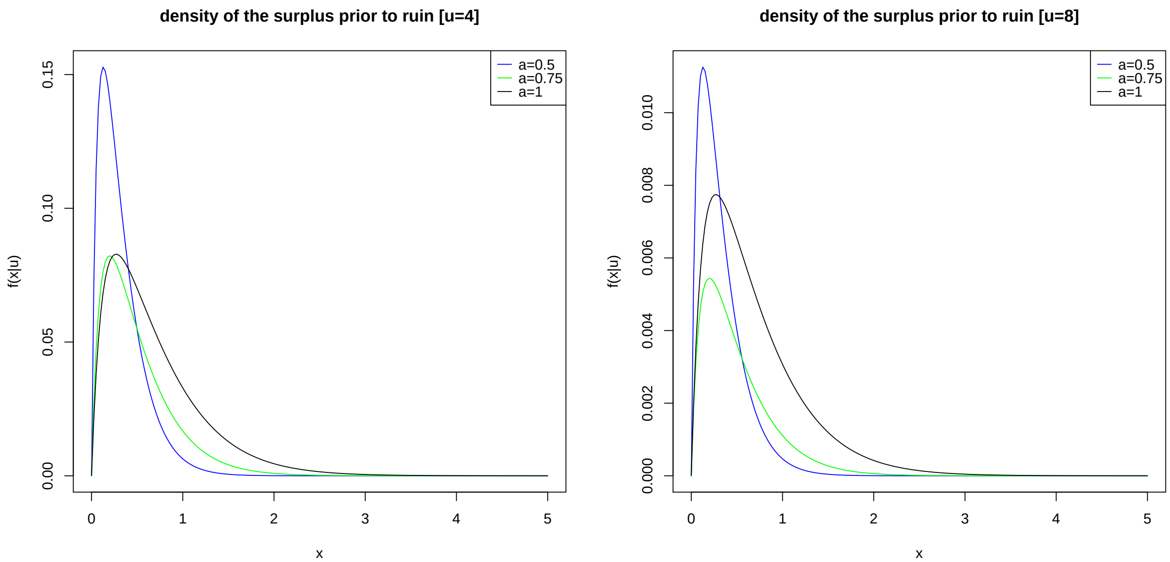

Figure 2.1: Graph of $x \mapsto f_a(x|u)$ when claim sizes are exponential

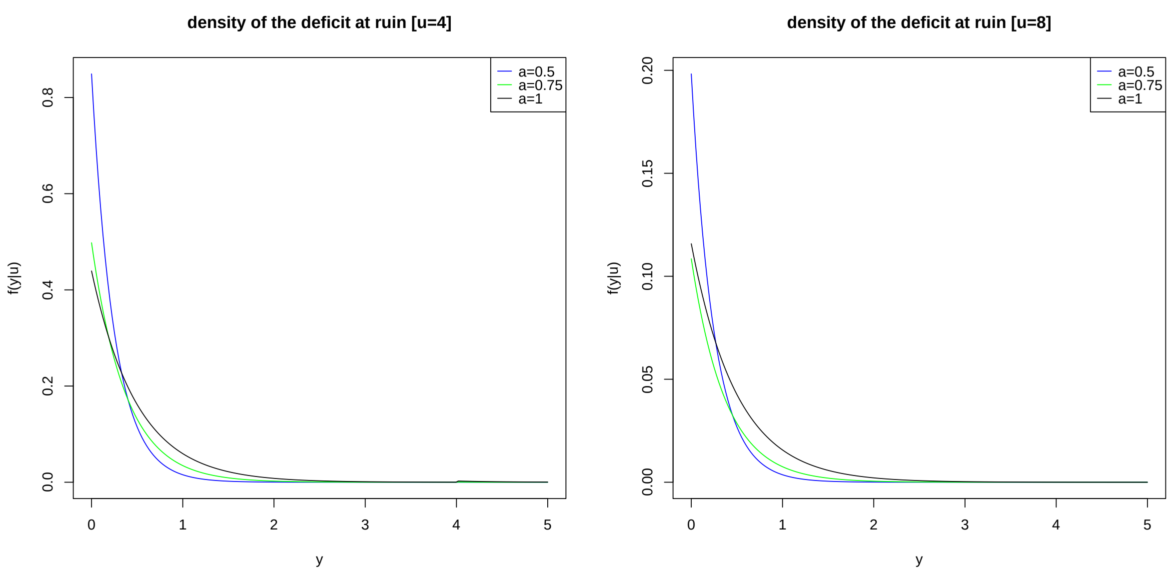

Figure 2.2: Graph of $y \mapsto f_a(y|u)$ when claim sizes are exponential

2.3.2 Hyper-exponentially distributed claim size

The study of the distribution of the deficit at ruin and the surplus prior to ruin can be carried out when there is an explicit expression of the ruin probability *. An explicit expression has been found when claim size is hyper-exponential (also called mixture of exponential), so we can study the impact of reinsurance.

*. cf. subsection 2.2.4

For the numerical applications, we will take the example of Actuarial mathematics *, where $f_X(x) = \frac{1}{2}\left(3e^{-3x} + 7e^{-7x}\right)$ and $\lambda = 3$. We know in this case that the roots of the Lundberg equation are rational †. And we have that the ruin probability is $\psi(u) = \frac{24}{35}e^{-x} + \frac{1}{35}e^{-6x}$ (when $\delta = 0$ and no reinsurance). With proportional reinsurance, we can solve the Lundberg equation (2.30). Thus, we have $\psi(u) = C_1 e^{-r_1 u} + C_2 e^{-r_2 u}$ with

$$C_1 = \frac{r_2(3-ar_1)(7-ar_1)}{21(r_2-r_1)} \text{ and } C_2 = \frac{r_1(7-ar_2)(3-ar_2)}{21(r_1-r_2)},$$

where the roots are

$$r_1 = \frac{-3+10\frac{C(a)}{a}+3\sqrt{\Delta_a}}{2C(a)}, \ r_2 = \frac{-3+10\frac{C(a)}{a}-3\sqrt{\Delta_a}}{2C(a)} \text{ and } \Delta_a = 1 + \left(4\frac{C(a)}{a\lambda}\right)^2.$$

Note that in this particular case, the Lundberg equation is a second order equation.

Hence, we have the following relation for the surplus prior to ruin

$$f_a(x|u) = f_a(x|0)\begin{cases} \frac{1-C_1e^{-r_1u}-C_2e^{-r_2u}}{1-C_1-C_2} & \text{if } x > u \geq 0 \\ \frac{C_1e^{-r_1(u-x)}+C_2e^{-r_2(u-x)}-C_1e^{-r_1u}-C_2e^{-r_2u}}{1-C_1-C_2} & \text{if } 0 < x \leq u \end{cases},$$

where $f_a(x|0) = \frac{3}{2C(a)}\left(e^{-3\frac{x}{a}} + e^{-7\frac{x}{a}}\right)$. The expression of the density of the deficit at ruin is a bit more complicated

$$f_a(y|u) = \begin{cases} \frac{3}{2C(a)}\left(e^{-3\frac{y}{a}} + e^{-7\frac{y}{a}}\right)\frac{1-C_1e^{-r_1u}-C_2e^{-r_2u}}{1-C_1-C_2} & \text{if } x > u \geq 0 \\ \frac{3}{2C(a)(1-C_1-C_2)}K(y) & \text{if } 0 < x \leq u \end{cases},$$

where

$$K(y) = \left(\frac{r_1C_1e^{-r_1u}e^{-3\frac{y}{a}}}{3-ar_1} + \frac{r_2C_2e^{-r_2u}e^{-3\frac{y}{a}}}{3-ar_2} + \frac{r_2C_2e^{-r_2u}e^{-7\frac{y}{a}}}{7-ar_2} + \frac{r_1C_1e^{-r_1u}e^{-7\frac{y}{a}}}{7-ar_1}\right).$$

The results are plotted in the graphs 2.3 and 2.4. The remarks of the previous subsection still apply. The reinsurance compress the density of ruin measures to its mean.

*. cf. example 12.10 of Bowers et al. (1986)

†. cf. Gerber & Dufresnes (1991*a*)

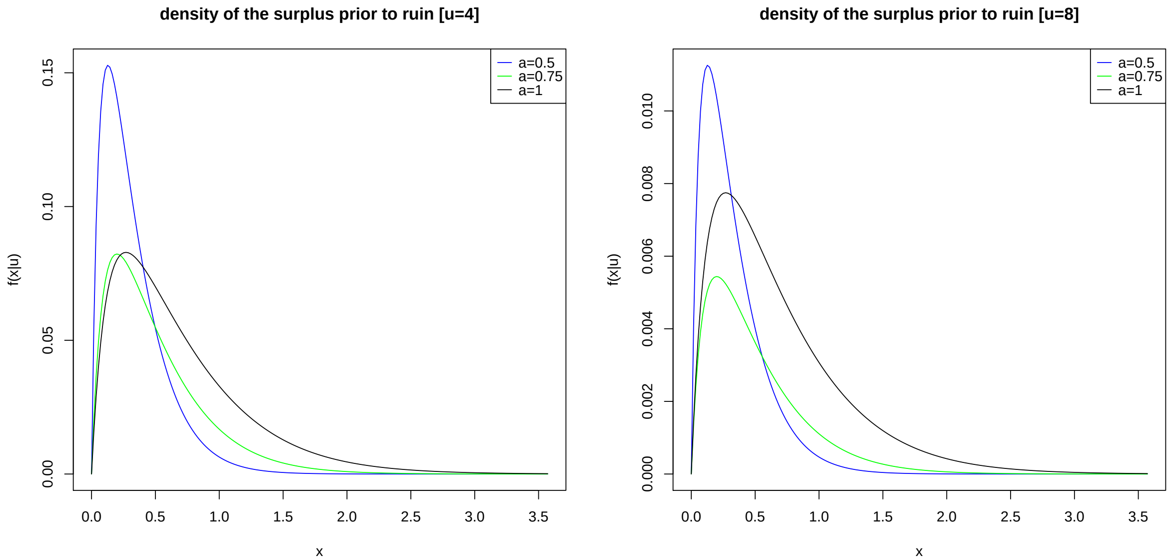

Figure 2.3: Graph of $x \mapsto f_a(x|u)$ when claim sizes are hyper-exponential

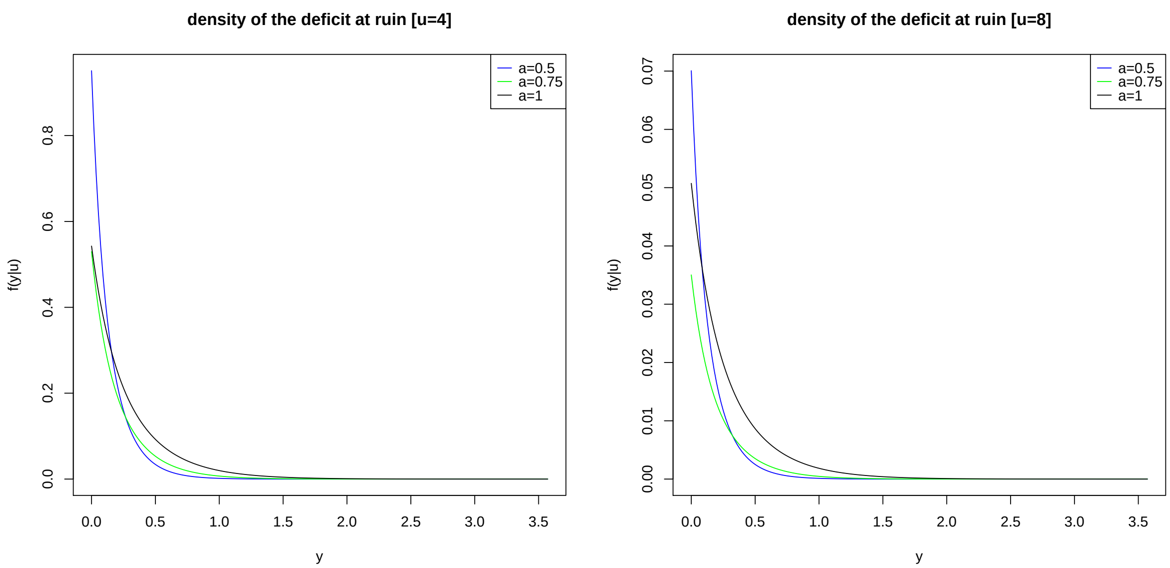

Figure 2.4: Graph of $y \mapsto f_a(y|u)$ when claim sizes are hyper-exponential

2.4 Consequences of reinsurance on the ruin probability

Let us study the effects of reinsurance on the ruin probability. As done in the previous section, we consider two claim size distribution: exponential $\mathcal{E}(1)$ * and a mixture of three exponentials $\mathcal{E}(0.46), \mathcal{E}(0.92)$ and $\mathcal{E}(1.38)$ with respective weights 0.1, 0.36 and 0.54 †. Expressions of ruin probabilities with proportional reinsurance when claim sizes are either exponential or hyper-exponential are derived in the subsections 2.2.6 and 2.2.7 (resp.).

The calculi have been done for two premium principles: the usual expected value and the standard deviation. The main difference between the "simple" expected value principle and the more sophisticated standard deviation principle is the latter depends on the tail of the tail distribution. In consequence, we will see the impact of the premium principles.

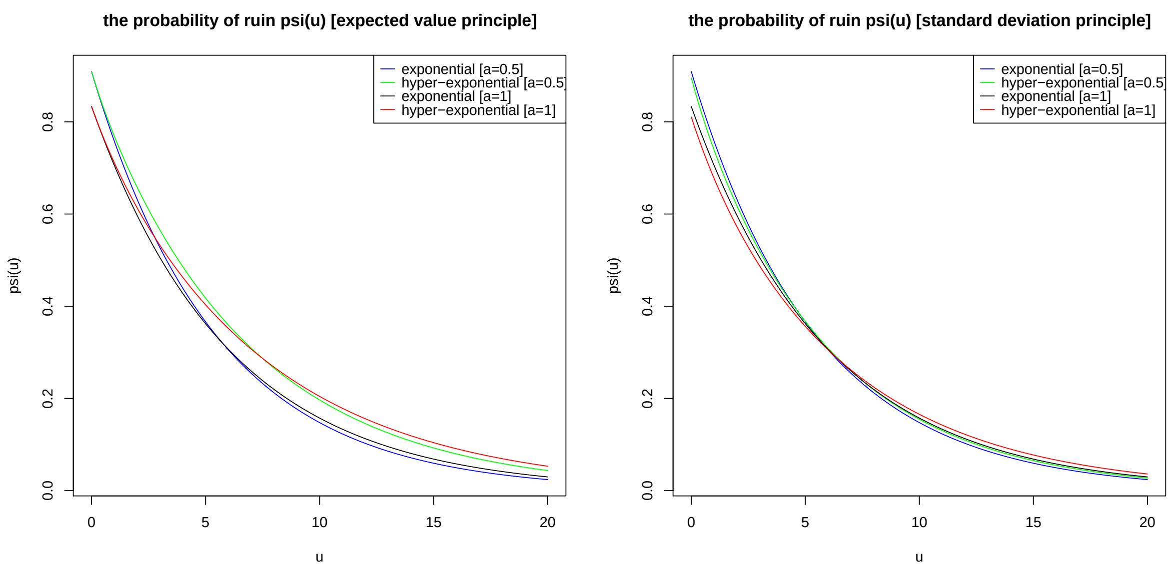

Figure 2.5: Graph of $u \mapsto \psi_a(u)$

In the figure 2.5, we see the ruin probability in zero $\psi_a(0)$ is greater with reinsurance $a = 0.5$ than without reinsurance $a = 1$. This is logic since the following relation obtained in section 2.2.4 holds

$$\psi_a(0) = \frac{a\lambda E[X]}{C(a)},$$

which is a decreasing function of a with the two considered premium principles.

Furthermore, the ruin probability with reinsurance falls stronger to 0 than without reinsurance. The functions $\psi_{1/2}$ downcrosses the functions ψ_1 around 6 for the exponential distribution and 7

∗. thus, the mean and the variance are 1

†. thus, the mean of the mixture is 1 and its variance 1.362949

for the mixture. Thus, beyond those values, the ruin probabilities are smaller with reinsurance than without.

Finally, we also see that the ruin probability is logically greater with the mixture of exponential distribution than with the exponential distribution since the variance is greater.

We can draw the same conclusions when using the standard deviation principle, but the effect of the claim size distribution is less striking. Indeed, with a premium principle taking the variance of the claim size distribution into account, such as the standard deviation, "heavy" tail distribution are more penalized than the exponential claim size case. Again, around the treshold 6, the ruin probability $\psi_{1/2}$ downcrosses the ruin probability ψ_1.

2.5 Conclusion

Along this chapter, we have studied the impact of proportional reinsurance on the Gerber-Shiu function, and its application on various ruin measures. We did numerical applications for three of those: the deficit at ruin, the surplus prior to ruin and the ruin probability, with the explicit expressions found when claim size distribution is either exponential or hyper-exponential.

We found that the distribution of the surplus prior to ruin is lower and less scattered with reinsurance than without. A similar effect is noticed on the distribution of deficit at ruin, whereas for the ruin probability, the reinsurance mitigates the risk only beyond a certain treshold.

Chapter 3

Explicit expressions of the ruin probability with phase-type distributions

In the ruin theory literature, the Sparre Andersen model had been studied in all its aspects. Asmussen & Rolski (1991) introduced phase-type distributions in the computation of ruin probabilities. Phase-type distributions come from the queuing theory, and are a general class of positive random variable distributions. They are part of the matrix exponential distributions. Asmussen & Rolski (1991) follows the work of Neuts, which applied queuing theory result in other fields. A complete review of phase-type distributions in the general area of risk theory was done Bladt (2005).

This section is dedicated to explicit expressions of the ruin probability in the Sparre Andersen model. So we work in a more restricted model than Chapter 1, since we assume the independence between claim sizes and waiting times. We use phase-type distributions in the computation of ruin probabilities for two purposes: (1) to implement ruin probabilities in the R package **actuar** [*] for a wide range of claim size distributions and (2) to analyze the effects of proportional reinsurance on ruin probabilities.

Phase-type distributions is a powerful tool to derive explicit expression of ruin probability in the Sparre Andersen model. This is mainly due to the fact that the aggregate claim size S_t is phase-type when claim sizes $(X_i)_i$ are phase-type.

We briefly recall the Sparre Andersen risk model:
- the claim arrival process $(N_t)_t$ is a renewal process;
- let $(W_i)_i$ be the sequence of i.i.d. [†] inter-occurence times [‡] and $(X_i)_i$ be the sequence of i.i.d. claim sizes. We suppose $\forall i, j \geq 0, X_i \perp W_j$;
- let u and C be resp. the initial capital and the premium rate. The surplus process is defined

∗. Goulet (2007)
†. independent and identically distributed
‡. we denote by G the distribution function of W_i.

by

$$U_t = u + Ct - \underbrace{\sum_{i=1}^{N_t} X_i}_{S_t};$$

– defining the time of ruin τ_u as the first time the surplus is strictly negative, the ruin probability is $\psi(u) = P(\tau_u < +\infty)$.

First, we present phase-type distributions, and then its application in ruin theory. Third, we briefly present the implementation of ruin probabilities in the R package **actuar** with phase-type distributions. Finally various numerical applications of ruin probabilities with proportional reinsurance are carried out.

3.1 Definition of phase-type distributions

A phase-type distribution $PH(\pi, T, m)$ (π a row vector of $\mathbb{R}^m$, T a $m \times m$ matrix) is defined as the distribution of the time to absorption in the state 0 of a Markov jump process, on the set $\{0, 1, \dots, m\}$, with initial probability $(0, \pi)$ and intensity matrix *

$$\Lambda = (\lambda_{ij})_{ij} = \left(\begin{array}{c|c} 0 & 0 \\ \hline t_0 & T \end{array}\right),$$

where the vector t_0 is $-T\mathbf{1}_m$ and $\mathbf{1}_m$ stands for the column vector of 1 in $\mathbb{R}^m$. This means that if we note $(M_t)_t$ the associated Markov process of a phase-type distribution, then we have

$$P(M_{t+h} = j / M_t = i) = \begin{cases} \lambda_{ij} h + o(h) & \text{if } i \neq j \\ 1 + \lambda_{ii} h + o(h) & \text{if } i = j \end{cases}.$$

The matrix T is called the sub-intensity matrix and t_0 the exit rate vector.

The cumulative distribution function of a phase-type distribution is given by

$$F(x) = 1 - \pi e^{Tx} \mathbf{1}_m,$$

and its density by

$$f(x) = \pi e^{Tx} t_0,$$

where e^{Tx} denote the matrix exponential defined as the matrix serie $\sum\limits_{n=0}^{+\infty} \frac{T^n x^n}{n!}$. The computation of matrix exponential is studied in details in subsection 3.4, but let us notice that when T is a diagonal matrix, the matrix exponential is the exponential of its diagonal terms.

*. matrix such that its row sums are equal to 0 and have positive elements except on its diagonal.

The moments of a phase-type distribution are given by $(-1)^n n! \pi T^{-n}\mathbf{1}$. Since phase-type distributions are platikurtic or light-tailed distributions, the Laplace transform exists

$$\widehat{f}(s) = \pi(-sI_m - T)^{-1} t_0,$$

where I_m stands for the $m \times m$ identity matrix.

One property among many is the set of phase-type distributions is dense with the set of positive random variable distributions. Hence, the distribution of any positive random variable can be written as a limit of phase-type distributions. However, a distribution can be represented (exactly) as a phase-type distribution if and only if the three following conditions are verified

- the distribution has a rational Laplace transform;
- the pole of the Laplace transform with maximal real part is unique;
- it has a density which is positive on $\mathbb{R}_+^\star$.

Here are some examples of distributions, which can be represented by a phase-type distribution

- exponential distribution $\mathcal{E}(\lambda)$: $\pi = 1$, $T = -\lambda$ and $m = 1$.
- generalized Erlang distribution $\mathcal{G}\left(n, (\lambda_i)_{1 \leq i \leq n}\right)$:

$$\pi = (1, 0, \dots, 0),$$

$$T = \begin{pmatrix} -\lambda_1 & \lambda_1 & 0 & \dots & 0 \\ 0 & -\lambda_2 & \lambda_2 & \ddots & 0 \\ 0 & 0 & -\lambda_3 & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & \lambda_{n-1} \\ 0 & 0 & 0 & 0 & -\lambda_n \end{pmatrix},$$

  and $m = n$.
- a mixture of exponential distribution of parameter $(p_i, \lambda_i)_{1 \leq i \leq n}$:

$$\pi = (p_1, \dots, p_n),$$

$$T = \begin{pmatrix} -\lambda_1 & 0 & 0 & \dots & 0 \\ 0 & -\lambda_2 & 0 & \ddots & 0 \\ 0 & 0 & -\lambda_3 & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & -\lambda_n \end{pmatrix},$$

  and $m = n$.
- a mixture of 2 (or k) Erlang distribution $\mathcal{G}(n_i, \lambda_i)_{i=1,2}$ with parameter p_i :

$$\pi = (\underbrace{p_1, 0, \dots, 0}_{n_1}, \underbrace{p_2, 0, \dots, 0}_{n_2}),$$

$$T = \begin{pmatrix} -\lambda_1 & \lambda_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & \ddots & \lambda_1 & 0 & \ddots & 0 & 0 \\ 0 & 0 & -\lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & -\lambda_2 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & \lambda_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda_2 \end{pmatrix},$$

and $m = n_1 + n_2$.

3.2 Ruin probability

In Asmussen & Rolski (1991), we have the following results: in the Cramér Lundberg model, when claim sizes are phase type $PH(\pi, T, m)$, the ruin probability is given by

$$\psi(u) = \pi_+ e^{Qx} \mathbf{1}_m, \tag{3.1}$$

with $Q = T + t_0\pi_+$ and $\pi_+ = -\frac{\lambda}{c}\pi T^{-1}$ where λ is the parameter of the Poisson process. In other words, the ruin probability is phase-type distributed $PH(\pi_+, Q, m)$.

In the Sparre Andersen model (i.e. arrival times have a general distribution G), the ruin probability is still phase type $PH(\pi_+, Q, m)$, but the sub-intensity matrix is the matrix solution of the fixed point equation

$$Q = \Phi(Q), \tag{3.2}$$

with $\Phi(K) = T + t_0\pi\hat{G}(K)$ and $\hat{G}(K) = \int_0^{+\infty} e^{Kx} G(dx)$; and the initial probability is given by

$$\pi_+ = \frac{\mathbf{1}^t(Q - T)}{c\mathbf{1}^t t_0},$$

where $\mathbf{1}^t$ denotes the tranpose of vector $\mathbf{1}$. The proofs of this result can be found in Asmussen (1992), mainly based on the fact that a geometric compound of phase-type distributions is still phase-type. The application of phase-type distributions in insurance are also well described in Hipp (2005).

3.3 Proportional reinsurance

We can extend the previous results, if we consider the insurer takes proportional reinsurance on his risk. As usual, we denote by $a \in]a_0, 1]$ (cf. the first part for the constant a_0) the retention rate, we have that the density of the risk $X(a)$ net of reinsurance is

$$f_{X(a)}(x) = \frac{1}{a} f_X\left(\frac{x}{a}\right).$$

If we suppose that X has a phase-type distribution $PH(\pi, T, m)$, we get

$$f_{X(a)}(x) = \frac{1}{a}\pi e^{\frac{Tx}{a}} t_0 = \pi e^{\frac{T}{a}x}(-\frac{T}{a}\mathbf{1}_m).$$

Hence, the random variable $X(a)$ is still phase-type distributed with parameters $PH(\pi, \frac{T}{a}, m)$. So, all the previous explicit expressions of ruin probability still hold with proportional reinsurance, we just have to change the subintensity matrix and the premium rate.

3.4 Computation of the ruin probability

The main problems of implementing this phase-type approach is to find Q in the Sparre Andersen model and to compute the exponential matrix e^{Qu} in both risk models.

Asmussen & Rolski (1991) eases the calculus of matrix Q in the Sparre Andersen model. If the distribution of inter-occurence times are phase-type $PH(\nu, S, n)$, then we have

$$\hat{G}(K) = (I_m \otimes \nu) (-K \oplus S)^{-1} (I_m \otimes s_0),$$

where $\otimes$ denotes the Kronecker product and $\oplus$ the Kronecker sum *. Therefore, the function Φ can be easily calculated.

However, there is an underlying issue when calculating the matrix Q as the fixed point solution of $Q = \Phi(Q)$ (with initial point T). We don't know if the function Φ is contractant. In the literature, we didn't find if it is true under certain conditions or wrong in all cases. So we can't use the Banach fixed point theorem †, which guarantee both the numerical stability and the convergence to the unique solution of (3.2). Note that the Banach fixed point theorem has the good quality to ensure the time of convergence to be exponential.

Now let us consider the problem of computing e^{Qu}. We recall that

$$e^{Qu} = \sum_{n=0}^{+\infty} \frac{Q^n u^n}{n!}.$$

There are various methods to compute the matrix exponential, Moler & Van Loan (2003) makes a deep analysis of the efficiency of different methods. In our case, we choose a decomposition method. We diagonalize the $n \times n$ matrix Q and use the identity

$$e^{Qu} = P e^{Du} P^{-1},$$

where D is a diagonal matrix with eigenvalues on its diagonal and P the eigenvectors. As we want to compute $\pi_+ e^{Qu} \mathbf{1}_m$ for different values of u. We compute

$$\psi(u) = \sum_{l=1}^{m} e^{\lambda_l u} \underbrace{\pi_+ P M_l P^{-1} \mathbf{1}_m}_{C_l},$$

where λ_i stands for the eigenvalues of Q, P the eigenvectors and $M_l = (\delta_{il}\delta_{lj})_{ij}$ (δ_{ij} is the symbol Kronecker, i.e. equals to zero except when $i = j$). As the matrix M_l is a sparse matrix with just a 1 on the l^{th} term of its diagonal. The constant C_i can be simplified. Indeed, if we denote by X_l the l^{th} column of the matrix P (i.e. the eigenvector associated to the eigenvalue λ_l) and Y_l the l^{th} row of the matrix P^{-1}, then we have

$$C_l \stackrel{\triangle}{=} \pi_+ P M_l P^{-1} \mathbf{1}_m = \pi_+ X_l \otimes Y_l \mathbf{1}_m.$$

Despite Q is not obligatorily diagonalizable, this procedure will often work, since Q may have a complex eigenvalue (say λ_i). In this case, C_i is complex but as $\psi(u)$ is real, we are ensured there

*. cf. appendix B.9
†. cf. appendix B.10

is $j \in [\![1, \ldots, m]\!]$, such that λ_j is the conjugate of λ_l. Thus, we get

$$e^{\lambda_i u} C_i + e^{\lambda_j u} C_j = 2cos(\Im(\lambda_i)u) e^{\Re\lambda_i u} \pi_+ \Re(X_i \otimes Y_i) \mathbf{1}_m - 2sin(\Im(\lambda_i)u) e^{\Re\lambda_i u} \pi_+ \Im(X_i \otimes Y_i) \mathbf{1}_m \in \mathbb{R},$$

where $\Re$ and $\Im$ stands resp. for the real and the imaginary part. And so we retrieve the fact that the ruin probability can be expressed as a sum of exponential and sinusoidal functions, which has been illustrated in Drekic et al. (2004). At the time, we are writing this memoir, we are currently working to compute matrix exponential when the matrix is not diagonalizable.

Finally, the ruin probability can be calculated by the function `ruinProb` of the package **actuar**. Details of usage can be found in appendix B.11, the usage will probably change in a very soon future. It is higly recommended to use the help directly in R with `help(ruinProb)`.

3.5 Numerical applications

For these numerical applications, we consider three cases of claim size distributions:

1. X follows an exponential distribution $\mathcal{E}(1)$, $E[X] = 1 = Var[X]$;
2. X is a mixture of exponential distribution $\mathcal{E}(0.46), \mathcal{E}(0.92)$ and $\mathcal{E}(1.38)$ with respective weights 0.1, 0.36 and 0.54. Thus, $E[X] = 1$ and $Var[X] = 1.362949$;
3. X follows a generalized Erlang* distribution with $\lambda_1 = \frac{10}{6}$, $\lambda_2 = \frac{10}{3}$ and $\lambda_3 = 10$. We have $E[X] = 1$ and $Var[X] = 0.46$.

We choose these three particular distribution in order to have the same expectation but different variances.

As for the inter-occurence time, we don't want to redo the exponential case (the Cramér-Lundberg model). That's why we choose an erlang distribution $\mathcal{G}(2, 1)$ and a hyper-exponential $\mathcal{E}(\frac{2}{3})$ and $\mathcal{E}(\frac{4}{9})$ with respective weights $\frac{1}{3}$ and $\frac{2}{3}$. They have the same mean, but different variances (resp. 2 and $\frac{4}{3}$).

The retention rate is either 1 (no reinsurance) or 0.5 (half of the risk is transferred to the reinsurer). And finally, we choose two premium principles, the expected value and the standard deviation. The definition can be found in section 1.1.3.

Result are plotted in the figures (3.1), (3.2), (3.3) and (3.4). Firstly, we notice that the ruin probabilities are lower with the Erlang(2) inter-occurence times than with the hyper-exponential. For instance, the ruin probability $\psi_a(0)$ are lower with the Erlang(2) distribution than with the hyper-exponential. And the tail of the ruin probability ψ_a decreases sharper with the Erlang(2) claim arrivals.

Secondly, the remarks on the impact of the retention rate on the ruin probability are not exactly the same: proportional reinsurance decreases the ruin probability $\psi_a(u)$ even for small values of initial capital u. Moreover, we can see that $\psi_{1/2}$ falls sharpier to 0 than ψ_1, except in the figure (3.4).

*. one way to characterize it is to say that it is a sum of independent but not identical exponential distributions.

Finally, the impact of the premium principle is the same as the previous numerical applications. The standard deviation principle reduces the differences of the ruin probability for the different claim size distributions.

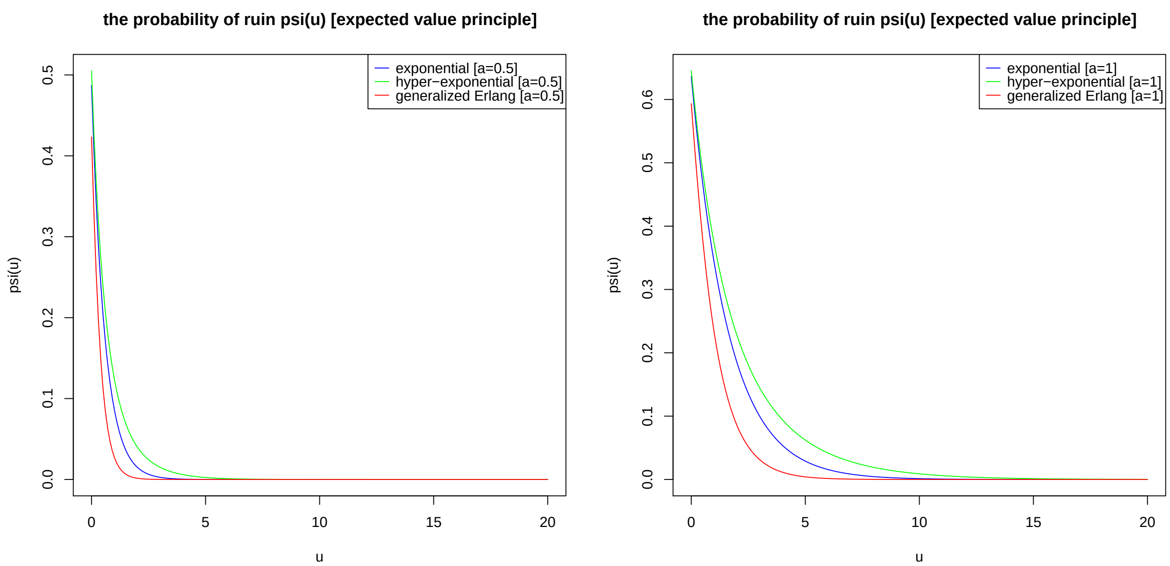

Figure 3.1: Graph of $u \mapsto \psi_a(u)$ when $W \sim \mathcal{G}(2,2)$ with the expected value principle

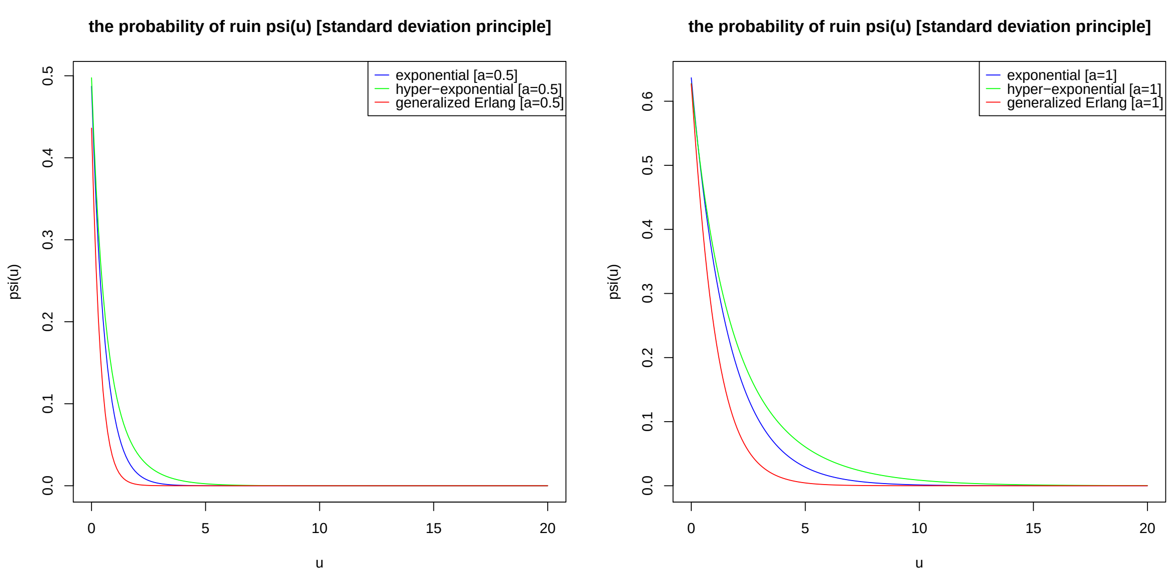

Figure 3.2: Graph of $u \mapsto \psi_a(u)$ when $W \sim \mathcal{G}(2,2)$ with the standard deviation

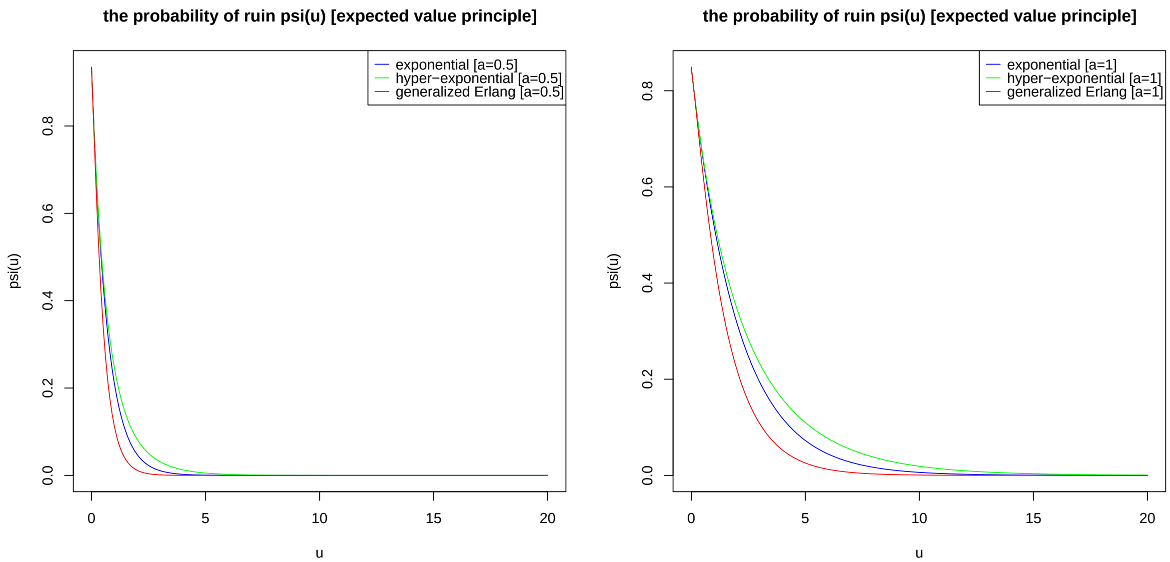

Figure 3.3: Graph of $u \mapsto \psi_a(u)$ when W is hyper-exponential with the expected value principle

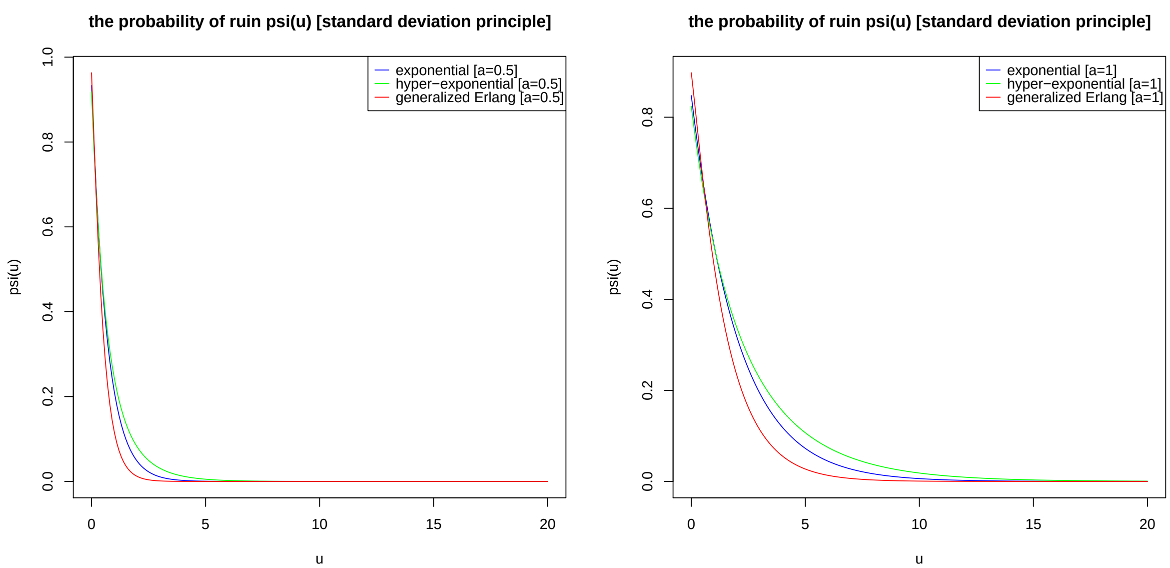

Figure 3.4: Graph of $u \mapsto \psi_a(u)$ when W is hyper-exponential with the standard deviation

3.6 Conclusion

This very short chapter studied the impact of proportional reinsurance and the inter-occurence distribution on the ruin probability in the Sparre Andersen model. The conclusions on the effects of reinsurance are almost the same as in the Cramér-Lundberg model: reinsurance mitigates the ruin probability for every initial capital. Furthermore, we have seen the more the inter-occurence distribution is "risky" (i.e. big variances), the more the ruin probability is small.

All this phase-type mechanic will be included in the next version of the R package **actuar** [*]. The new version will provide functions to compute the density, the distribution function, the moments, etc... for the phase-type distribution in addition to the function `ruinProb` for the ruin probability through phase-type distributions. Another function computing the adjustment coefficient will be added, hence all the numerical applications on the adjustment coefficient in a context of dependence of Chapter 1 could be done with the packages **actuar** and **copula** [†].

*. Goulet (2007)
†. Yan & Kojadinovic (2007)

Conclusion

In this memoir, we dealed with three topics linked to reinsurance and ruin theory. We covered various subjects such the Gerber-Shiu with proportional reinsurance, optimal reinsurance according to the adjustment coefficient and phase-type distributions to compute ruin probabilities.

First, we studied the adjustment coefficient with reinsurance in a context of dependence between claim severity and claim frequency. The optimization of the adjustment coefficient with respect to retention parameter raised the underlying issue of its unimodality. We showed it is always ensured with proportional reinsurance. However for excess of loss reinsurance, a condition has to be imposed in order to have unimodality. Second, introducing proportional reinsurance in the Gerber-Shiu function (in the Cramér-Lundberg model) leaded to interesting conclusions. With proportional reinsurance, the distributions of the deficit at ruin and the surplus prior to ruin are compressed to their mean. Concerning the ruin probability, proportional reinsurance minimizes the risk only beyond a certain treshold of capital. Third, we have seen that phase-type distributions ease the calculation of ruin probabilities in the Sparre Andersen model. We presented the implementation of those computations in the R package **actuar** as well as the impact of proportional reinsurance on ruin probabilities.

Each of the three different chapters leaves many questions open for further research. For instance, it remains to study a combination of excess of loss and proportional reinsurance, as well as non constant risk margins η and η_R. Two obvious extensions to Chapter 2 would be excess of loss reinsurance and analysis of the Gerber-Shiu function in the Sparre Andersen model (based on Gerber & Shiu (2005)). Finally, one could compare approximations of ruin probabilities with phase-type distributions and the Beekman's formula (cf. Beekman's Convolution formula in Kaas (2006)), for heavy-tailed claim size distribution.

In conclusion, we could consider optimal reinsurance in a dynamic setting, such that Schmidli (2001) did with proportional reinsurance in the Cramér-Lundberg model. Phase-type distributed inter-occurence times could also be analyzed in the Gerber-Shiu function.

Appendices

Appendix A

Optimal Reinsurance in a Context of Dependence

A.1 Proof: $\frac{\partial^2 h}{\partial r^2}(r,a) < 0$

We recall that:

$$\frac{\partial h}{\partial r}(r,a) = \frac{E\left[(aX - C(a)W)e^{r(aX-C(a)W)}\right]}{E\left[e^{r(aX-C(a)W)}\right]}.$$

So, we have:

$$\frac{\partial^2 h}{\partial r^2}(r,a) = \frac{E\left[(aX - C(a)W)^2 e^{r(aX-C(a)W)}\right]}{E\left[e^{r(aX-C(a)W)}\right]} - \left(\frac{E\left[(aX - C(a)W)e^{r(aX-C(a)W)}\right]}{E\left[e^{r(aX-C(a)W)}\right]}\right)^2. \quad \text{(A.1)}$$

(A.1) is positive since it is a variance of an Esscher transform. Therefore, the function $r \mapsto h(r,a)$ is convex, since the function h is C^2 on $\mathbb{R}^+$.

In the case of excess of loss reinsurance, we have the function h is defined as

$$h(r,L) = \ln\left(E[e^{r(X\wedge L - C(L)W)}]\right).$$

The analogy of equation (A.1) for excess of loss reinsurance is

$$\frac{\partial^2 h}{\partial r^2}(r,L) = \frac{E\left[(X\wedge L - C(L)W)^2 e^{r(X\wedge L-C(L)W)}\right]}{E\left[e^{r(X\wedge L-C(L)W)}\right]} - \left(\frac{E\left[(X\wedge L - C(L)W)e^{r(X\wedge L-C(L)W)}\right]}{E\left[e^{r(X\wedge L-C(L)W)}\right]}\right)^2,$$

which is again positive. So the function $r \mapsto h(r,L)$ is convex, using the same argument as the previous demonstration.

A.2 Admissibility condition on 'a'

We want to solve

$$aE[X] - C(a)E[W] = 0,$$

where $C(a)$ is given by

$$\frac{E(X)}{E(W)}(\eta - \eta_R + a(1+\eta_R)).$$

Thus the equation above becomes

$$a\frac{E(X)}{E(W)} = \frac{E(X)}{E(W)}(\eta - \eta_R + a(1+\eta_R)) \Leftrightarrow -a\eta_R = \eta - \eta_R.$$

Hence, it yields to

$$a = \frac{\eta_R - \eta}{\eta_R}.$$

A.3 Sufficient condition for unimodality

Let us show the following proposition for $\phi : \mathbb{R} \mapsto \mathbb{R}$

Proposition. *If ϕ is a C^2 function, ϕ is an unimodal function on I if the equation $\phi'(t) = 0$ has a unique root $t^\star$, such as $\phi''(t^\star) < 0$.*

Proof. As ϕ'' is a continuous function (since ϕ is C^2), it exists $\epsilon > 0$, such as $\forall t \in]t^\star - \epsilon, t^\star + \epsilon[$, $\phi''(t) < 0$. This implies that ϕ' is a strictly decreasing function on $]t^\star - \epsilon, t^\star + \epsilon[$, which cancels in $t^\star$.
As ϕ' is a continuous function and $\phi'(t) = 0$ has a unique root on I, $\phi'(t)$ is strictly positive on $I \cap]-\infty, t^\star[$ and strictly negative on $I \cap]t^\star, +\infty[$. Otherwise, ϕ' would cancel more than once. Hence, ϕ is strictly increasing on $I \cap] - \infty, t^\star[$, reaches its maximum on $t^\star$ and then is strictly decreasing $I \cap]t^\star, +\infty[$. □

Note that the condition "the equation $\phi'(t) = 0$ has a unique root $t^\star$" is crucial. If the latter equation has multiple roots, we are not ensured that ϕ is unimodal on I, since $t^\star$ may only be on a local maximum. An easy counter-example of non-unimodality is when $\phi(t) = t\sin(t)$ on $\mathbb{R}$.

A.4 Implicit function theorem

Theorem 1. *Let F be a bivariate C^1 function on some open disk with center in (a, b), such that $F(a, b) = 0$. If $\frac{\partial F}{\partial y}(a, b) \neq 0$, then there exists an $h > 0$, and a unique function φ defined for $]a - h, a + h[$, such that*

$$\varphi(a) = b \quad \textit{and} \quad \forall |x - a| < h, F(x, \varphi(x)) = 0.$$

Moreover on $|x - a| < h$, the function φ is C^1 and

$$\varphi'(x) = - \left. \frac{\frac{\partial F}{\partial x}(x, y)}{\frac{\partial F}{\partial y}(x, y)} \right|_{y=\varphi(x)}.$$

A.5 $a \mapsto f(a)$ has a unique root

We recall that f is defined as

$$a \mapsto E\left[(X - C'(a)W)e^{R(X(a) - C(a)W)}\right].$$

Thus the first derivative with respect to a is given by

$$f'(a) = RE\left[(X - C'(a)W)^2 e^{R(X(a)-C(a)W)}\right] + R'(a)E\left[(X - C'(a)W)(X(a) - C(a)W)e^{R(X(a)-C(a)W)}\right].$$

Thus we have, $f'(a) > 0$ when $f(a) = 0 \Leftrightarrow R'(a) = 0$.

Furthermore, we have that $f(a_0) = -\eta_R E[X] < 0$ and $f(1) > 0$. Indeed,

$$f(1) = E\left[(X - (1+\eta_R)\frac{E[X]}{E[W]}W)e^{R(X-(1+\eta)\frac{E[X]}{E[W]}W)}\right] > E\left[(X - (1+\eta)\frac{E[X]}{E[W]}W)e^{R(X-(1+\eta)\frac{E[X]}{E[W]}W)}\right].$$

The right-hand side of the previous inequality has the same sign as $\frac{\partial h}{\partial r}(R, 1)$, which is positive, as we have already seen. This implies that f is contiunous function which is strictly increasing each time it crosses the abcisse line, such that $f(a_0) < 0$ and $f(1) > 0$. So f cancels exactly once.

In figure $(A.1)$, there are some examples of the so called f function for the different distributions used in numerical applications.

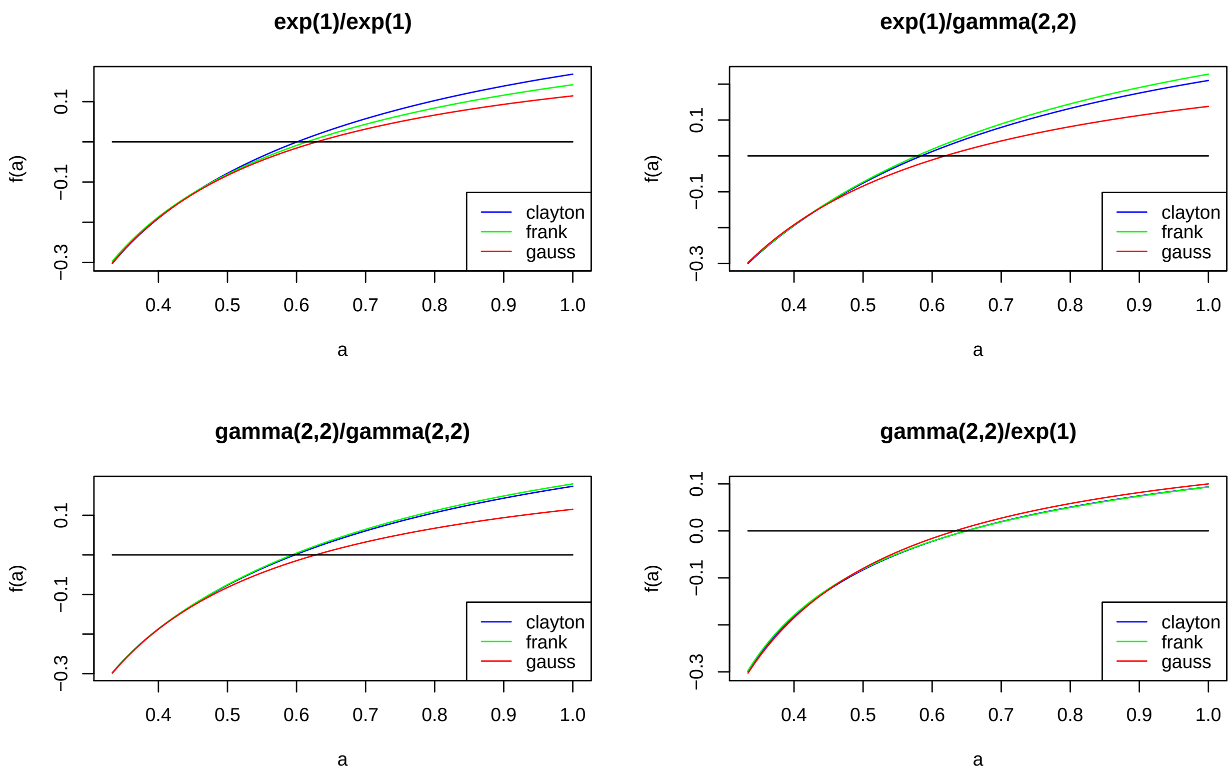

Figure A.1: Graph of $a \mapsto f(a)$

A.6 Proof: 'g' is a decreasing function with exponential premiums

We want the sign of $Cov(X, e^{kX})$, with X a positive random variable and k a positive real. Let ϕ and φ be the functions xe^{kx} and e^{kx} respectively. These two functions are convex on $\mathbb{R}_+^\star$ since they are C^2, $\phi''(x) = (2k + x^2)e^{kx} > 0$ and $\varphi''(x) = k^2 e^{kx}$. Thus, we have the minoration

$$Cov(X, e^{kX}) \triangleq E[Xe^{kX}] - E[X]E[e^{kX}] \geq E[X]e^{kE[X]} - E[X]e^{kE[X]} = 0,$$

using the Jensen inequality, $E[\Phi(X)] \geq \Phi(E[X])$ for a convex function Φ.

A.7 Proof: properties of $X \wedge L$ as a function of L

We have

$$X \wedge L = \begin{cases} X & \text{if } X \leq L \\ L & \text{if } X > L \end{cases}.$$

So differentiating with respect to L, we get

$$\frac{\partial X \wedge L}{\partial L}(L) = \begin{cases} 0 & \text{if } X \leq L \\ 1 & \text{if } X > L \end{cases} = \mathbf{1}_{(X>L)}.$$

Let us study the derivative of $\mathbf{1}_{(X>L)}$ with respect to L. The indicator function is differentiable on $\mathbb{R}_+$ except on X. Indeed, we have

$$\lim_{L \to X^-} \frac{\mathbf{1}_{(L>L)} - \mathbf{1}_{(X>L)}}{L - X} = \lim_{L \to X^-} \frac{-1}{L - X} = +\infty,$$

and

$$\lim_{L \to X^+} \frac{\mathbf{1}_{(L>L)} - \mathbf{1}_{(X>L)}}{L - X} = \lim_{L \to X^+} 0 = 0.$$

So $\frac{\partial \mathbf{1}_{(X>L)}}{\partial L} \overset{a.s.}{=} 0$ since X is continuous.

A.8 $L \mapsto f(L)$ has multiple roots

In the case of excess of loss reinsurance, f is defined as

$$f(L) = E\left[(\mathbf{1}_{(X>L)} - C'(L)W)e^{R(X(L) - C(L)W)}\right].$$

Let us notice $\lim\limits_{L \longrightarrow +\infty} f(L) = 0$ since both functions $\mathbf{1}_{(X>L)}$ and $C'(L) = (1 + \eta_R)\frac{\bar{F}_X(L)}{E[W]}$ tends to null. But the solution $L = +\infty$ is not a solution mathematically and in practice. Because this involves that the insurer takes no reinsurance at all.

We also have

$$f(L_0) \triangleq E\left[(\mathbf{1}_{(X>L)} - C'(L)W)e^0\right] = -\eta_R \bar{F}_X(L_0) < 0.$$

Let us study the first derivative of f

$$f'(L) = -E\left[C''(L)We^{R(X(L)-C(L)W)}\right] + RE\left[(\mathbf{1}_{(X>L)} - C'(L)W)^2 e^{R(X(L)-C(L)W)}\right] + R'(L)E\left[(\mathbf{1}_{(X>L)} - C'(L)W)(X(L) - C(L)W)e^{R(X(L)-C(L)W)}\right].$$

We implicitly supposed that X is continuous, otherwise $C''(L)$ is not defined since the density of X is used. Furthermore, in this case, $C''(L) = -(1 + \eta_R)\frac{f_X(L)}{E[W]} < 0$. The problem is f is not an increasing function. Thus, it is difficult to be sure that f has one root.

In figure (A.3), there are some examples of the so called f function for the different distributions used in numerical applications, with the expected value principle. The corresponding $L \mapsto R(L)$ graphs with these particular marginals can be found in section 1.3.3-"Unimodality". Note that f has multiple roots in `gamma(100,100)/gamma(2,2)`.

In figures (A.2) and (A.4), we plotted the f function with respectively the exponential and the standard deviation premium calculation principle. As we can see, the graph reveals that f may have an "asymptotic" root ($+\infty$) or one root ($< +\infty$). The corresponding $L \mapsto R(L)$ graphs can be found in section 1.3.3-"The impact of the premium principles".

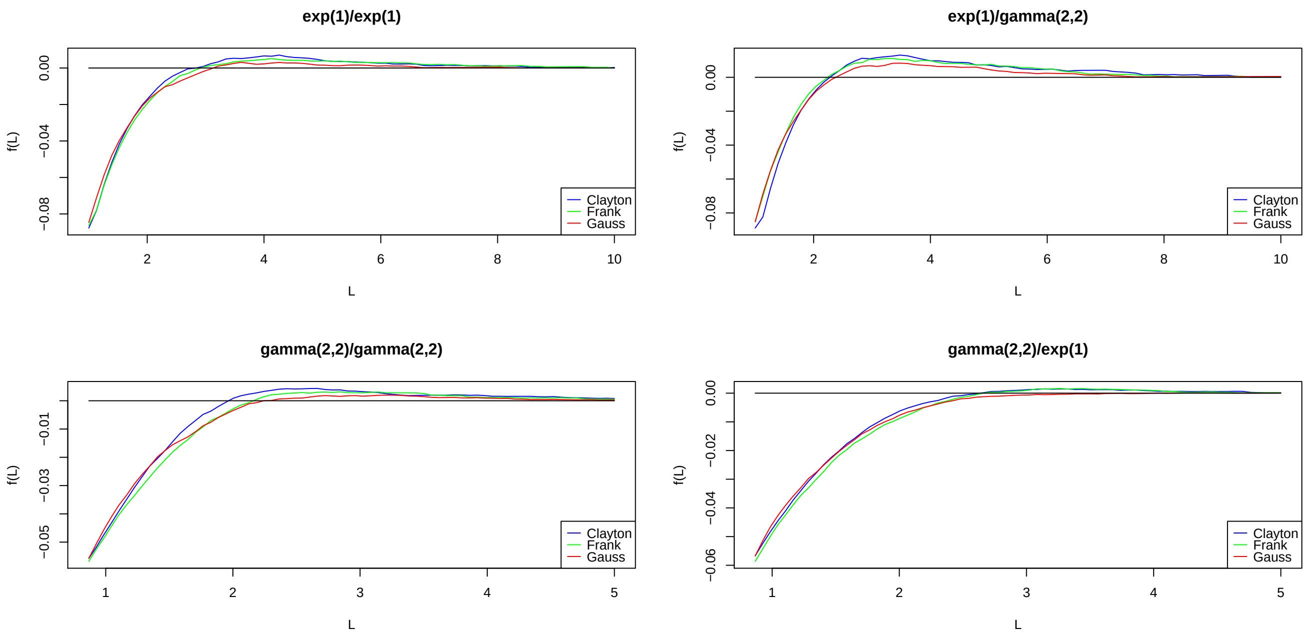

Figure A.2: Graph of $L \mapsto f(L)$ with the exponential principle

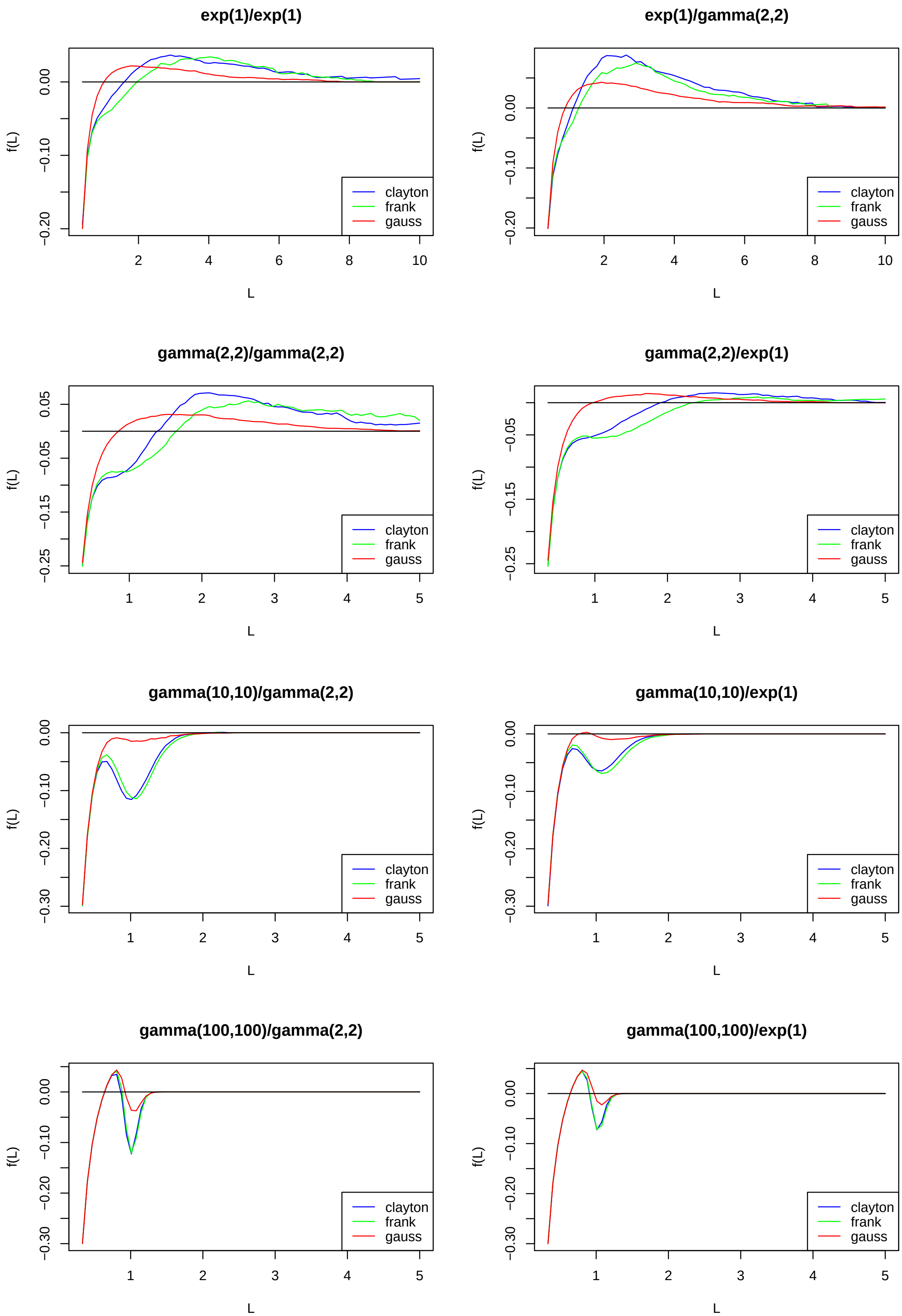

Figure A.3: Graph of $L \mapsto f(L)$ with the expected value principle

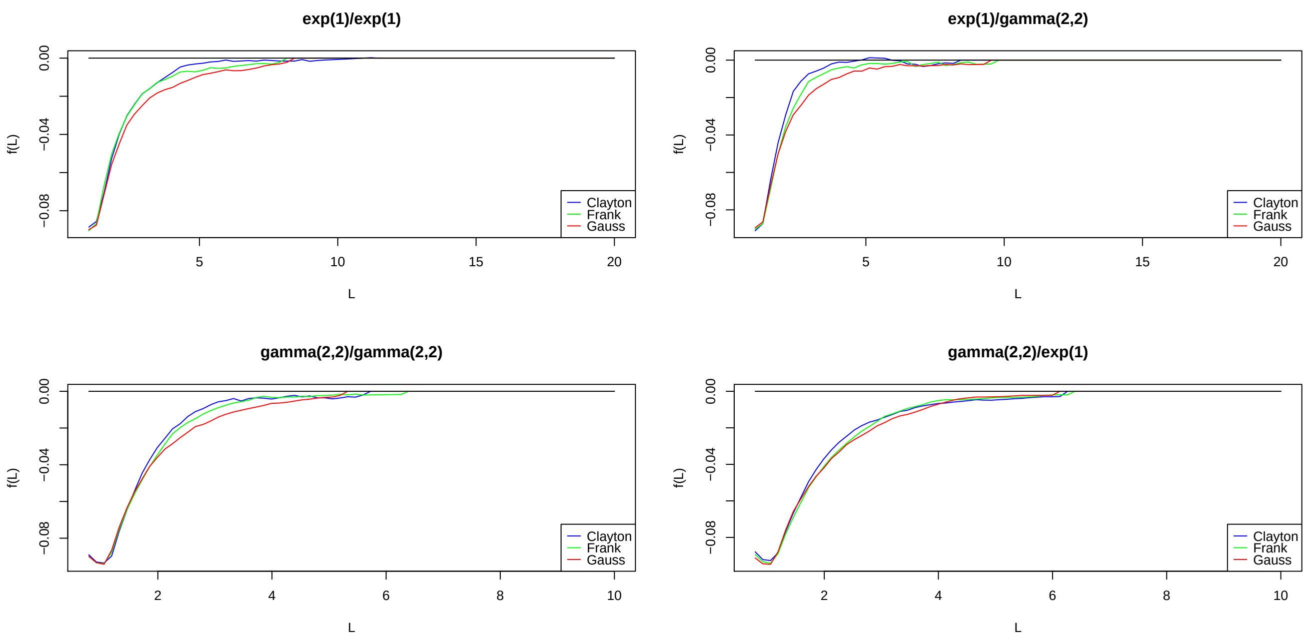

Figure A.4: Graph of $L \mapsto f(L)$ with the standard deviation principle

A.9 Truncated moment generating function

We study the truncated moment generating function defined as (X follows a gamma distribution $\mathcal{G}(\alpha, \lambda)$)

$$M_{X\wedge L}(t) \stackrel{\triangle}{=} \int_0^{+\infty} e^{tx\wedge L} f_X(x)dx = \int_0^L e^{tx} f_X(x)dx + \int_L^{+\infty} e^{tL} f_X(x)dx = \int_0^L e^{tx} f_X(x)dx + e^{tL}\overline{F}_X(L,\alpha,\lambda)$$

$$= \int_0^L e^{tx} \frac{x^{\alpha-1}\lambda^\alpha e^{-\lambda x}}{\Gamma(\alpha)} dx + e^{tL}\overline{F}_X(L,\alpha,\lambda) = \frac{\lambda^\alpha}{(\lambda-t)^\alpha} \int_0^L \frac{x^{\alpha-1}(\lambda-t)^\alpha e^{-(\lambda-t)x}}{\Gamma(\alpha)} dx + e^{tL}\overline{F}_X(L,\alpha,\lambda).$$

Hence, we have

$$M_{X\wedge L}(t) = M_X(t,\alpha,\lambda) F_Y(L,\alpha,\lambda-t) + e^{tL}\overline{F}_X(L,\alpha,\lambda),$$

where Y follows a gamma distribution $\mathcal{G}(\alpha, \lambda - t)$).

Appendix B

Consequences of reinsurance

B.1 Comment by Dickson (1998)

Dickson (1998) proposed a new way to get the expression (2.7). The functional equation (2.3) becomes when taking its Laplace transform

$$\widehat{\varphi}_\delta(\xi)\xi - \varphi_\delta(0) = \frac{\delta+\lambda}{C}\widehat{\varphi}_\delta(\xi) - \frac{\lambda}{C}\widehat{\varphi}_\delta(\xi)\widehat{f}_X(\xi) - \frac{\lambda}{C}\widehat{\omega}(\xi),$$

thus

$$\widehat{\varphi}_\delta(\xi) = \frac{\lambda\widehat{\omega}(\xi) - C\varphi_\delta(0)}{\delta+\lambda-C(a)\xi-\lambda\widehat{f}_X(\xi)},$$

which is equivalent to (2.7) since $C\varphi_\delta(0) = \lambda\widehat{\omega}(\rho)$ and ρ verifies the Lundbeg equation (2.5). When supposing that the penalty function $w(x, y) = 1$, Dickson finds (2.8).

B.2 Key renewal theorem

The Key Renewal theorem

Theorem 2. *Consider the integral equation $Z = f * Z + z$. If we have R such that $\widehat{f}(-R) = 1$ (i.e. the function $x \mapsto e^{Rx} f(x)$ is a density), then we have*

$$Z(x) \underset{+\infty}{\sim} \frac{\widehat{z}(-R)}{-\left(\widehat{f}\right)'(-R)} e^{-Rx}.$$

This version of the key renewal theorem deals with defective or excessive renewal equation.

B.3 Definition of a martingale

Definition. *Let $(X_t)_t$ be a continuous process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. $(X_t)_t$ is a $\mathcal{F}_t$-martingale if*

- *$(X_t)_t$ is $\mathcal{F}_t$-adapted, i.e. $\forall t > 0, (X_t)_t$ is $\mathcal{F}_t$-mesurable;*
- *$(X_t)_t$ is integrable, i.e. $\forall t > 0, E[|X_t|] < +\infty$;*
- *$\forall t > s, E[X_t/\mathcal{F}_s] = X_s$.*

In general, the filtration $\mathcal{F}_t$ is the natural filtration of the process $(X_t)_t$, i.e. $\sigma(X_t)$.

B.4 Explanations on the process V_ξ

The first two conditions of the definition of a martingale are verified by $(V_{\xi,t})_t$. Since, the integrability condition is

$$E[|V_{\xi,t}|] = E\left[e^{-\delta t+\xi U_t}\right] \leq e^{-\delta t+\xi(u+Ct)} < +\infty,$$

the second condition is verified. The first one is also verified for $\mathcal{F}_t = \sigma(S_t)$, because $(V_{\xi,t})_t$ is a continuous composition (exponential functional) of mesurable process : the compound Poisson process S_t.

B.5 Explanations on the optional sampling theorem and its application

The Doob's Optional Sampling theorem

Theorem 3. *Let $(X_t)_t$ be a martingale on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and σ, τ two bounded stopping times, such that $\sigma \overset{a.s.}{\leq} \tau$. Then*

$$E[X_\tau/\mathcal{F}_\sigma] \overset{a.s.}{=} X_\sigma.$$

We used this theorem in two different cases. First, $X_t = V_{-R,t}$, $\sigma = 0$ and $\tau = \tau \wedge n = \min(\tau, n)$. Hence, we have

$$e^{-Ru} = E\left[e^{-\delta\tau\wedge n - RU_{\tau\wedge n}}/U_0 = u\right].$$

Since $\forall t > 0, e^{-\delta\tau\wedge n-RU_{\tau\wedge n}} < 1$ (hence the right-hand side of the previous relation converges when $n \to +\infty$), we have just to tend n to $+\infty$ ($\tau \wedge n \to \tau$) to get the expected relation. Secondly, we use the theorem with $X_t = V_{\rho,t}$, $\sigma = 0$ and $\tau = T_x \wedge n = \min(T_x, n)$. Thus, we obtain

$$e^{\rho u} = E\left[e^{-\delta T_x\wedge n+\rho U_{T_x\wedge n}}/U_0 = u\right].$$

Since $\forall t > 0, e^{-\delta T_x\wedge n-RU_{T_x\wedge n}} < e^{\rho x}$, the right-hand side of the previous relation converges when $n \to +\infty$. Hence

$$e^{\rho u} = E\left[e^{-\delta T_x+\rho U_{T_x}}/U_0 = u\right].$$

With proportional reinsurance, we use the same reasoning.

B.6 Inverse Laplace transform with the Heaviside's expansion formula

We recall the definition of the Laplace transform:

Definition. *Let f be a piecewise continuous function. The Laplace transform of f is the unique function $\widehat{f}$ defined by*

$$\widehat{f}(s) = \int_0^{+\infty} e^{-st} f(t)dt.$$

The Laplace transform is an application $L : f \mapsto \widehat{f}$, also written $\mathcal{L}(f)$. Note that the sectionally continuousness of f is a sufficient condition for existence of Laplace transform.

The Laplace transform has many properties. Some of them are listed here : linearity, first and second translation, change of scale. And also the derivation property $\widehat{f^{(n)}}(s) = s^n \widehat{f}(s) - \sum_{i=0}^{n-1} s^i f^{(n-1-i)}(0)$, the integral property $\mathcal{L}\left(t \mapsto \int_0^t f(u)du\right)(s) = \frac{\widehat{f}(s)}{s}$, the initial value $f(+\infty) = \lim_{s\to 0} s\widehat{f}(s)$ if $f(+\infty)$ exists, and the final value $f(0) = \lim_{s\to+\infty} s\widehat{f}(s)$.

The inverse Laplace transform:

Definition. *Let F be a continuous function. The inverse Laplace transform of F is the unique function f such that*

$$\mathcal{L}(f) = F.$$

The inverse Laplace transform is also written $\mathcal{L}^{-1}(f)$.

The inverse Laplace transform has the corresponding properties of the Laplace transform, such as linearity, first and second translation, change of scale. But also the derivation property $\mathcal{L}^{-1}(f^{(n)})(t) = (-1)^n t^n L^{-1}(f)(t)$, the integral property $\mathcal{L}^{-1}\left(s \mapsto \int_s^{+\infty} f(u)du\right)(s) = \frac{L^{-1}(f)(t)}{t}$.

If we take the Laplace transform of the exponential, we have

$$f(t) = e^{at} \quad \text{and} \quad \widehat{f}(s) = \frac{1}{s-a}, s > a.$$

Whence the inverse of Laplace transform of $F(s) = \frac{1}{s-a}$ is $L^{-1}(F)(t) = e^{at}$. One obvious way to find the inverse of Laplace transform of a fraction is to do a partial fraction expansion, such that

$$F(s) = \sum_{i=1}^{n} \frac{A_i}{s-\alpha_i} \;\Leftrightarrow\; \mathcal{L}^{-1}(F)(t) = \sum_{i=1}^{n} A_i e^{\alpha_i t},$$

where $(\alpha_i)_{1\leq i\leq n}$ are the roots of the denominator of F.

The Heaviside Expansion Formula is the application of this principle:

Proposition. *Let P and Q be two polynoms such that deg$(P) \leq$ deg$(Q) = n$ (i.e. the degree of P is not bigger than the degree Q). If Q has n distinct roots $(\alpha_i)_{1\leq i\leq n}$, then it follows*

$$\mathcal{L}^{-1}\left(\frac{P}{Q}\right)(t) = \sum_{i=1}^{n} \frac{P(\alpha_i)}{Q'(\alpha_i)} e^{\alpha_i t}.$$

If the degree of P is strictly bigger than the degree of Q, then the fraction can be written as $\frac{P(s)}{Q(s)} = R(s) + \frac{P_1(s)}{Q(s)}$ with $deg(P_1) \leq deg(Q)$.

If the roots of the denominator of the fraction F are not simple, say the root r of multiplicity m, we have

$$F(s) = \frac{N(s)}{(s+r)^m} = \sum_{i=1}^{m} \frac{B_i}{(s-r)^i},$$

then

$$\mathcal{L}^{-1}(F)(t) = \sum_{i=1}^{m} \beta_i \frac{t^{i-1}}{(i-1)!} e^{rt},$$

where

$$\beta_{m-i} = \frac{1}{i!} \lim_{s \to r} \frac{\partial^i \left(s \mapsto \frac{P(s)(s-r)^m}{Q(s)} \right)}{\partial s^i}(s).$$

Note that if there are some complex roots, the trigonometric functions make their appearance in the inverse Laplace transform, complete results on the Laplace transform can be found in Spiegel (1965).

B.7 Derivative of a function defined as a integral

Theorem 4. *Let a function $f : X \times [a,b] \mapsto \mathbb{R}$, and 2 functions $u, v : X \mapsto [a,b]$, with the set $X \subset \mathbb{R}$. We suppose that f is $C^{2\,*}$ on $X \times [a,b]$, and u, v are C^1 on X. Then the function ϕ defined as*

$$\phi(x) = \int_{u(x)}^{v(x)} f(x,t)dt$$

is C^1 on X and its derivative is

$$\phi'(x) = \int_{u(x)}^{v(x)} \frac{\partial f}{\partial x}(x,t)dt + v'(x)f(x,v(x)) - u'(x)f(x,u(x)).$$

B.8 Relations between $f_a(x, y|0)$ and $f_a(x, y|u)$

This section of the appendix briefly recalled the main points of the demonstration of the relation between $f(x|0)$ and $f(x|u)$, that is presented in Gerber & Shiu (1998). By T_a (resp. T_b), we denote the stopping times defined as the first time upcrosses the level a (b) with $a \leq u < b$. If the surplus starts above the "barrier level" a (b), the process will have to drop below a (b) and then upcross the barrier.

Let $T_{a,b}$ be $T_a \wedge T_b = \min(T_a, T_b)$ the minimum of the two stopping times. Then we define the "complementary" functions

$$A(a,b|u) = E\left[e^{-\delta T_{a,b}} \mathbf{1}_{(U_{T_{a,b}}=a)}/U_0 = u\right] = E\left[e^{-\delta T_a} \mathbf{1}_{(T_a<T_b)}/U_0 = u\right],$$

and

$$B(a,b|u) = E\left[e^{-\delta T_{a,b}} \mathbf{1}_{(U_{T_{a,b}}=b)}/U_0 = u\right] = E\left[e^{-\delta T_b} \mathbf{1}_{(T_a>T_b)}/U_0 = u\right].$$

$*$. twice differentiable with continuous second derivative

This two functions have some interesting properties

1. $A(a,b|u) + B(a,b|u) = E\left[e^{-\delta T_{a,b}}/U_0 = u\right]$
2. for all constant k, $A(a,b|u) = A(a+k, b+k|u+k)$ and $B(a,b|u) = B(a+k, b+k|u+k)$

For $a' < a \leq u < b < b'$, the authors of Gerber & Shiu (1998) derive the following system

$$\begin{cases} A(a,b'|u) & = A(a,b|u) + B(a,b|u)A(a,b'|b) \\ B(a',b|u) & = A(a,b|u)B(a',b|a) + B(a,b|u) \end{cases},$$

then taking $a' = +\infty$, $a = 0$, $b = x$ and $b' = +\infty$, they solve this linear system. Finally, they get

$$\begin{cases} A(0,x|u) & = \frac{e^{\rho x}\psi(u) - e^{\rho u}\psi(x)}{e^{\rho x} - \psi(x)} \\ B(0,x|u) & = \frac{e^{\rho u} - \psi(u)}{e^{\rho x} - \psi(x)} \end{cases}.$$

At last using that $f(x|u) = \frac{f(x|0)}{B(0,u|0)}$ (because if ruin occurs with a surplus equal to x before the ruin, then the surplus must have cross $u < x$), they obtain the first part of the relation between $f(x|0)$ and $f(x|u)$ (i.e. when $0 \leq u < x$).

To show the second part of the relation (when $x \leq u$), they use duality on the process $U_t^\star$ defined as U_t if ruin never occurs, otherwise, $-U_{T_0 - t}$ for $0 \leq t \leq T_0$ and U_t for $t > T_0$. T_0 is the time of recovery, since it is defined as the time where the surplus first upcrosses 0 (which implies ruin has occured). From this transformation, they derived an equality between $B(0,u|0)$ and $A(-u,0|-x)$, from which they derived the second part. When introducing proportional reinsurance, it does not affect the proof, since the surplus process U_t^a is still a linear combinaison of a coumpound Poisson process.

B.9 Kronecker product and sum

The Kronecker product $A \otimes B$ is defined as the $mn \times mn$ matrix

$$A \otimes B = (A_{i_1,j_1} B_{i_2,j_2})_{i_1 i_2, j_1 j_2},$$

when A is a $m \times m$ matrix of general term $(A_{i_1,j_1})_{i_1,j_1}$ and B a $n \times n$ matrix of general term $(B_{i_2,j_2})_{i_2,j_2}$. Note that the Kronecker can also be defined for non-square matrixes.

The Kronecker sum $A \oplus B$ is given by the $mn \times mn$ matrix

$$A \otimes B = A \otimes I_m + B \otimes I_n,$$

where I_m and I_n are the identity matrixes of size m and n. This definition is right only for square matrixes A and B.

B.10 Banach fixed point theorem

The Banach Fixed Point theorem

Theorem 5. *Let E be a complete normed vector space (i.e. a Banach space) and $f : E \mapsto E$ be a continuous function. If there exists $0 < k < 1$, such that $\forall (x,y) \in E^2$,*

$$||f(x) - f(y)|| < k||x - y||,$$

(i.e. f is contractant), then there is a unique fixed point $x^\star \in E$, such that

$$f(x^\star) = x^\star.$$

So any sequence $(f(x_n))_n$ will converge exponentially to the fixed point $x^\star$, since

$$||x^\star - x_n|| < \frac{k^n}{1-k}||x_1 - x_0||.$$

B.11 Function `ruinprob` in actuar

It is higly recommended to use the help directly in R with `help(ruinProb)`. The current help is:

Description:
Compute the infinite time ruin probability in the model of Cramér-Lundberg or Sparre Andersen, using the results of Gerber-Dufresnes (1988) and Asmussen-Rolski (1991).

Usage:
`ruinProb(model="CramerLundberg",param,premRate)`

Arguments:
`model` a string indicating the model used, either Cramér-Lundberg (default value) or Sparre Andersen
`param` a list with the following components: either `lambda`, `pi`, `T` and `m` in the CramÈr-Lundberg model or `nu`, `S`, `n`, `pi`, `T`, `m` in the Sparre Andersen model; where `lambda` is the Poisson process parameter (of the claim arrival process), `pi,T,m` the parameters of the claim size phase-type distribution and `nu,S,n` the parameters of the inter-occurence times phase-type distribution
`premRate` the premium rate (which must respect the positive safety loading constraint)
Value:
Function `ruinProb` computes the ruin probability and returns the ruin probability as a function of one parameter, the initial capital.

Author:
Vincent Goulet and Christophe Dutang

References:
Asmussen, S. and Rolski, T. (1991). **Computational methods in risk theory: A matrix algorithmic approach**. Insurance: Mathematics and Economics, 10:259-274

Gerber, H. U. and Dufresnes, F. (1991). **Three methods to calculate the probability of ruin**. Astin Bull., 19(1):71-90

Examples:

```
## Cramer Lundberg - exponential claim sizes
    resExpPsi <- ruinProb("CramerLundberg", list( lambda=3 ,pi=1, T=2, m=1),
     1.1*3/2)

    gridU <- seq(0,5/2,length.out=100) #vector of initial capitals
    resExp <- resExpPsi(gridU)

## Cramer Lundberg - mixture of 3 exponential claim sizes
## E(1), E(3) and E(6) with respective weights 1/3, 1/2, 1/6
    Lambda <- 3
    matT <- array(c(-1,0,0,0,-3,0,0,0,-6),c(3,3))
```

```
    weight <- c(1/3,1/2,1/6)
    beta <- c(1,3,6)
    premiumRate <- sum(weight/beta)*Lambda*1.1

    resMixExpPsi <- ruinProb("CramerLundberg",
     list( lambda=Lambda ,pi=poids, T=matT, m=3), premiumRate)

    gridU <- seq(0,5*sum(weight/beta),length.out=100)#vector of initial capitals
    resMixExp <- resMixExpPsi(gridU)

## Sparre Andersen - Dickson (1992) numerical applications
## Dickson, D. (1992). On the distribution of surplus prior to ruin.
## North American Actuarial Journal
    S <-array(c(-1,0,0,1,-1,0,0,1,-1),c(3,3))
    probnu <- c(.4,.2,.4)

    matT <- array(c(-1,0,0,0,1,-3,0,0,0,3,-2,0,0,0,2,-4),c(4,4))
    probpi <- c(.2,.3,.4,.1)

    resErlangPsi <- ruinProb("SparreAndersen",
                 list( pi=probpi,T=matT,m=length(probpi),
                  nu=probnu,S=S,n=length(probnu) ), 1)

    gridU <- seq(0,5*16/15,length.out=100)
    resErlang <- resErlangPsi(gridU)
```

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